

1. Assume $(\Omega, \mathcal{B})$ is a measurable space, μ and ν are two equivalent σ -finite measures on it, and $1 \leq p < \infty$. Show that the map

$$L^p(\Omega, d\mu) \rightarrow L^p(\Omega, d\nu), \quad f \mapsto \left(\frac{d\mu}{d\nu}\right)^{1/p} f,$$

is an isometric isomorphism.

2. Assume X and Y are Banach spaces and we have decompositions $X = \bigoplus_{i=1}^n X_i$ and $Y = \bigoplus_{i=1}^m Y_i$ into direct sums of closed subspaces. Then linear operators $T: X \rightarrow Y$ can be viewed as matrices of operators $(T_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$, where $T_{ij}: X_j \rightarrow Y_i$.

(1) Show that we have decompositions $X^* = \bigoplus_{i=1}^n X_i^*$ and $Y^* = \bigoplus_{i=1}^m Y_i^*$, where X_i^* is identified with the subspace of functionals $f \in X^*$ that vanish on X_j for $j \neq i$. Show then that the transpose T^t has matrix $(T_{ji}^t)_{1 \leq i \leq n, 1 \leq j \leq m}$.

(2) Show that the composition rule for matrices of operators is exactly the same as for usual matrices, that is, if we have another Banach space $Z = \bigoplus_{i=1}^k Z_i$ and an operator $S = (S_{ij})_{i,j}: Y \rightarrow Z$, then the matrix of ST is $(\sum_{l=1}^m S_{il} T_{lj})_{i,j}$.

3. Show that if X and Y are finite-dimensional vector spaces, then every linear operator $T: X \rightarrow Y$ has index $\dim X - \dim Y$. Therefore Fredholm index is interesting only for operators on infinite-dimensional spaces.

4. A (cochain) *complex of vector spaces* is a sequence of vector spaces $(X_n)_n$, where n runs through an interval of $\mathbb{Z}$, together with linear operators $d_n: X_n \rightarrow X_{n+1}$ such that $d_n d_{n-1} = 0$ for all n (for which both operators are defined), that is, $\text{Im } d_{n-1} \subset \ker d_n$. The n -th *cohomology* of such a complex is defined as the quotient space

$$H^n(X_\bullet, d_\bullet) = \ker d_n / \text{Im } d_{n-1}.$$

Generalizing a result proved in the lectures, show that if

$$0 \rightarrow X_0 \xrightarrow{d_0} X_1 \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} X_n \rightarrow 0$$

is a complex of finite-dimensional vector spaces, then

$$\sum_{k=0}^n (-1)^k \dim X_k = \sum_{k=0}^n (-1)^k \dim H^k(X_\bullet, d_\bullet).$$