

Suggested solution to the exam in MAT4410, December 17, 2012.

Problem 1. Let λ denote Lebesgue measure on $X = [0, \infty)$. Find the limit

$$\lim_{n \rightarrow \infty} \int_{X \times X} e^{-(\frac{x^2 y^2}{n} + x + y)} d(\lambda \otimes \lambda)(x, y).$$

Solution: We have $e^{-(\frac{x^2 y^2}{n} + x + y)} \rightarrow e^{-(x + y)}$ as $n \rightarrow \infty$ pointwise on $X \times X$. By Tonelli's theorem, which applies since all the functions involved are measurable and non-negative, it follows that

$$\int_{X \times X} e^{-(x + y)} d(\lambda \otimes \lambda)(x, y) = \left(\int_X e^{-x} d\lambda(x) \right) \left(\int_X e^{-y} d\lambda(y) \right),$$

and since $\int_X e^{-x} d\lambda(x) = \lim_n \int_{[0, n]} e^{-x} dx = 1$ e.g. by Monotone Convergence theorem, it follows that $e^{-(x + y)} \in L^1(X \times X)$. Then

$$\lim_{n \rightarrow \infty} \int_{X \times X} e^{-(\frac{x^2 y^2}{n} + x + y)} d(\lambda \otimes \lambda)(x, y) = 1$$

by Dominated Convergence Theorem.

Problem 2. Let c_0 denote the Banach space of sequences converging to zero. Let $a = \{a_n\}_{n \geq 1}$ be a sequence of complex numbers such that $\sum_{n=1}^{\infty} a_n b_n$ is convergent for every $\{b_n\}_{n \geq 1} \in c_0$. For every $k \geq 1$, define $T_k : c_0 \rightarrow \mathbb{C}$ by $T_k(b) = \sum_{j=1}^k a_j b_j$ for $b = \{b_n\}_{n \geq 1}$ in c_0 .

2a. Show that T_k is a bounded linear functional for every $k \geq 1$.

2b. Conclude that $a \in l^1(\mathbb{N})$. What is the relationship between $\|a\|_1$ and $\|T_k\|$ for $k \geq 1$?

Solution for 2a: since $b_n \rightarrow 0$, the sequence $b = \{b_n\}_{n \geq 1}$ is bounded. Then

$$|T_k(b)| \leq \sum_{j=1}^k |a_j b_j| \leq \|b\|_{\infty} \sum_{j=1}^k |a_j|.$$

Hence $\|T_k\| \leq \sum_{j=1}^k |a_j|$, so T_k is bounded (you need to fill in the details for proving that T_k is linear).

Solution for 2b: fix $b = \{b_n\}_{n \geq 1}$ in c_0 . Then

$$\lim_{k \rightarrow \infty} T_k(b) = \lim_{k \rightarrow \infty} \sum_{j=1}^k a_j b_j \leq \sum_{j=1}^{\infty} a_j b_j < \infty.$$

Thus by the Banach-Steinhaus theorem, the map $Tb = \lim_{k \rightarrow \infty} T_k(b)$ defines a bounded operator $T : c_0 \rightarrow \mathbb{C}$. Moreover, $\|T\| = \sup_{k \geq 1} \|T_k\|$. Let $\alpha_j \in \mathbb{C}$ such that $a_j \alpha_j = |a_j|$ for every $j \geq 1$. Since

$$T_k(\alpha_1, \dots, \alpha_k, 0 \dots) = \sum_{j=1}^k a_j \alpha_j = \sum_{j=1}^k |a_j|,$$

it follows that $\|T_k\| = \sum_{j=1}^k |a_j|$ for $k \geq 1$. Hence $\|a\|_1 = \sum_{n=1}^{\infty} |a_n| = \|T\|$, so that $a \in l^1(\mathbb{N})$.

Problem 3. 4a. Formulate a consequence of the Hahn-Banach extension theorem for linear functionals appropriate for linear subspaces of normed spaces.

Solution. If X is a normed space, then a bounded linear functional l on a linear subspace Y admits an extension $\bar{l}$ to a bounded linear functional with $\|l\| = \|\bar{l}\|$ (choose the convex function $\phi(\alpha x) = \|x\||\alpha|$ as a bound for l).

3b. If X is a normed space and $Y \subset X$ is a linear subspace, show that Y is dense in X if and only if the only element $\varphi \in X^*$ such that $\varphi(y) = 0$ for all $y \in Y$ is the zero functional $\varphi_0(x) = 0$ for all $x \in X$.

Solution: Suppose first Y is dense. Let $\varphi \in X^*$ such that $\varphi(y) = 0$ for all $y \in Y$. We must show that $\varphi(x) = 0$ for all x . If there is $x \in X$ such that $\varepsilon_0 = |\varphi(x)| > 0$, choose $y \in Y$ with $\|x - y\| < \varepsilon_0 / \|\varphi\|$. Then $|\varphi(x)| = |\varphi(x - y)| \leq \|\varphi\| \cdot \|x - y\| < \varepsilon_0$, a contradiction. Thus $\varphi = \varphi_0$. For the converse direction, if there is $x \in X \setminus \overline{Y}$, then $d = \text{dist}(x, Y) > 0$, and by a consequence to the Hahn-Banach theorem for normed spaces there is $\varphi \in X^*$ with $\varphi(x) = d$ and $\varphi(y) = 0$ for all $y \in Y$. Thus $\varphi \neq \varphi_0$, a contradiction to the assumption.

3c. Suppose that μ is a Borel measure on $[0, 1]$ such that $\int_{[0,1]} x^k d\mu(x) = 0$ for all $k \geq 1$. Show that $\mu = 0$.

Solution: (Note that in the hypothesis one should assume $k \geq 0$ in order to avoid complications.) Let $X = C[0, 1]$ with the supremum norm, and let Y be the subspace of polynomials in one variable. It is known that Y is dense in X . By the Riesz Representation theorem, there is a bounded functional $\varphi(f) = \int_{[0,1]} f d\mu$ on X . The assumption that $\varphi(x^k) = 0$ for all $k \geq 0$ implies by continuity of φ that φ vanishes on all elements of Y . Then φ is the zero functional by 3b, so $\mu([0, 1]) = \|\varphi\| = 0$, and therefore $\mu = 0$.

3d. (This problem is independent of problems 3b and 3c.) Let λ be the Lebesgue measure on $[0, 1]$ and let $L^p([0, 1], \lambda)$ for $1 < p < \infty$ be the Banach space of p -integrable, complex-valued functions on $[0, 1]$. Suppose that $\{f_k\}_{k \geq 1}$ is a sequence of elements in $L^p([0, 1], \lambda)$ and $\{a_k\}_{k \geq 1}$ is a sequence of

complex numbers for which there exists $C > 0$ such that

$$|\sum_{k=1}^m \alpha_k a_k| \leq C \left(\int_{[0,1]} |\sum_{k=1}^m \alpha_k f_k(x)|^p d\lambda(x) \right)^{1/p}$$

for any choice of complex numbers $\alpha_1, \dots, \alpha_m$ and any $m \geq 1$. Let $\frac{1}{p} + \frac{1}{q} = 1$. Show that there is $g \in L^q([0, 1], \lambda)$ such that

$$\int_{[0,1]} f_k(x) g(x) d\lambda(x) = a_k, \forall k \geq 1.$$

Solution: Let $Y = \text{span}\{f_k \mid k \geq 1\}$. Define $\varphi_0 : Y \rightarrow \mathbb{C}$ by

$$\varphi_0\left(\sum_{k=1}^m \alpha_k f_k\right) = \sum_{k=1}^m \alpha_k a_k$$

for any choice of complex numbers $\alpha_1, \dots, \alpha_m$ and any $m \geq 1$. This is a linear map, and the assumption gives that φ_0 is bounded with $\|\varphi_0\| \leq C$. By the Hahn-Banach theorem there is an extension $\varphi \in L^p([0, 1])^*$. Since $L^p([0, 1])^* \cong L^q([0, 1])$, there is a function $g \in L^q([0, 1])$ such that

$$\varphi(f) = \int_{[0,1]} f g d\lambda$$

for every $f \in L^p([0, 1])$. Hence $\int_{[0,1]} f_k(x) g(x) d\lambda(x) = \varphi_0(f_k) = a_k$ for all $k \geq 1$, as claimed.

Problem 4. Let (X, Σ) be a measurable space and λ, μ be measures on Σ with λ finite, μ σ -finite and $\lambda \ll \mu$. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\mu(A) < \delta \Rightarrow \lambda(A) < \varepsilon, \forall A \in \Sigma.$$

Solution: If $f = d\lambda/d\mu$, then let $X_n = \{x \in X \mid |f(x)| \leq n\}$ for every $n \geq 1$. Proceed as in the proof of Lemma 9.19 (Teschl): first $\lambda(X \setminus X_n) \rightarrow 0$, so there is n such that $\lambda(X \setminus X_n) < \varepsilon/2$. Take now $\delta = \varepsilon/(2n)$.

4b. Let $X = \mathbb{R}$ with its Borel σ -algebra $\mathcal{B}$ and let λ be Lebesgue measure on $\mathcal{B}$. Define a measure on $\mathcal{B}$ by

$$\mu(A) = \int_A \frac{1}{e^{|x|}} d\lambda(x) \text{ for all } A \in \mathcal{B}.$$

Explain why $\lambda \ll \mu$. Solution: first note that $\mu \ll \lambda$ by construction, because $1/e^{|x|}$ is measurable and non-negative. Since $h = d\mu/d\lambda$ is the function

$h(x) = e^{-|x|}$, which satisfies $h(x) > 0$ for all x , we have by a known result that $1/h$ is the Radon-Nikodym derivative $d\lambda/d\mu$.

4c. With $(\mathbb{R}, \mathcal{B})$, λ and μ as in question 4b, show that the conclusion of problem 4a does not hold.

Solution: Let $\varepsilon = 1$. Let $\delta > 0$ arbitrary, and choose $n \in \mathbb{N}$ such that $1/e^n < \delta$. Let $A = [n, n+1]$. For $x \in A$, we have $e^x \geq e^n$, so $h(x) \leq 1/e^n < \delta$ on A . Thus

$$\mu(A) = \int_{[n, n+1]} \frac{1}{e^{|x|}} d\lambda(x) < \int_{[n, n+1]} \delta d\lambda(x) = \delta \lambda(A) = \delta,$$

but $\lambda(A) \geq 1$.