EXERCISES 10

Excercise 1 (Properties of conditional typical sequences)**.** Consider a joint probability distribution $p_{AB} \in \mathcal{P}(\Sigma_A \times \Sigma_B)$ and let $\mathcal{T}_{n,\epsilon}(p_{AB}|x_1,\ldots,x_n)$ denote the ϵ-typical strings conditioned on $(x_1,\ldots,x_n) \in \Sigma_A^n$ as defined in the lecture. Show the following:

(1) For all $\epsilon > 0$ we have

$$\lim_{n\to\infty} \sum_{x_1,\ldots,x_n \in \Sigma_A^n} \sum_{y_1,\ldots,y_n \in \mathcal{T}_{n,\epsilon}(p_{AB}|x_1,\ldots,x_n)} p_{AB}(x_1,y_1)\cdots p_{AB}(x_n,y_n) = 1.$$

Hint: Consider the function $f : \Sigma_A \times \Sigma_B \to [0,\infty)$ defined by

$$f(x,y) = \begin{cases} -\log(p(x,y)) + \log(p_A(x)) & \text{if } p(x,y) > 0 \\ 0 & \text{otherwise,} \end{cases}$$

and consider the random variable $f(X,Y)$ where (X,Y) is distributed according to p_{AB}. Compute the expected value of $f(X,Y)$, and apply the weak law of large numbers.

(2) For all $n \in \mathbb{N}$ and all $\epsilon > 0$ we have

$$\sum_{x_1,\ldots,x_n \in \Sigma_A^n} p_A(x_1)\cdots p_A(x_n)|\mathcal{T}_{n,\epsilon}(p_{AB}|x_1,\ldots,x_n)| < 2^{n(H(p_{AB})-H(p_A)+\epsilon)}.$$

Excercise 2 (Properties of conditional typical projections)**.** For any ensemble $\{p_A(x),\rho_x\}_{x\in\Sigma_A}$ let $\Lambda_{x_1,\ldots,x_n,\epsilon}$ denote the projection onto the ϵ-typical subspace conditioned on $x_1,\ldots,x_n$. Show the following:

(1) For all $\epsilon > 0$ we have

$$\lim_{n\to\infty} \sum_{x_1,\ldots,x_n \in \Sigma_A^n} p_A(x_1)\cdots p_A(x_n)\langle \Lambda_{x_1,\ldots,x_n,\epsilon}, \rho_{x_1}\otimes\cdots\otimes\rho_{x_n}\rangle_{HS} = 1.$$

(2) For all $n \in \mathbb{N}$ and all $\epsilon > 0$ we have

$$\sum_{x_1,\ldots,x_n \in \Sigma_A^n} p_A(x_1)\cdots p_A(x_n)\operatorname{Tr}\left[\Lambda_{x_1,\ldots,x_n,\epsilon}\right] < 2^{n\left(\sum_{x\in\Sigma_A} p_A(x)H(\rho_x)+\epsilon\right)}.$$

Excercise 3. Without using the HSW-theorem, prove that

$$C(T^{\otimes k}) = kC(T),$$

for any $k \in \mathbb{N}$ and any quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$.

Excercise 4 (Holevo information of quantum channels)**.** Consider the Holevo information $\chi(T)$ of a quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$. We will show that

$$\chi(T) = \chi\left(\{p(x), T(\rho_x)\}_{x\in\Sigma}\right),$$

for some ensemble $\{p(x),\rho_x\}_{x\in\Sigma}$ of quantum states $\rho_x \in D(\mathcal{H}_A)$, a probability distribution $p \in \mathcal{P}(\Sigma)$ for an alphabet Σ satisfying $|\Sigma| \leqslant \dim(\mathcal{H}_A)^2$.

(1) Consider a quantum state $\sigma \in D(\mathcal{H})$ and fix a finite alphabet Σ. Show that the set $\mathcal{CQ}_\Sigma(\sigma)$ of classical-quantum states

(1) $$\rho_{CQ} = \sum_{x\in\Sigma} p(x)|x\rangle\langle x| \otimes \rho_x,$$

for $p \in \mathcal{P}(\Sigma_N)$ and $\rho_x \in D(\mathcal{H})$ such that $\rho_Q = \sum_{x\in\Sigma} p(x)\rho_x = \sigma$ is compact and convex. What about the set $\mathcal{CQ}_\Sigma$ of classical-quantum states of the form (1) without any restriction on the marginal state ρ_Q?

(2) Show that any extreme point ρ_{CQ} of the set $\mathcal{CQ}_\Sigma(\sigma)$ has the form (1) with $|\{x \in \Sigma_N \ : \ p(x) \neq 0\}| \leqslant \mathrm{rk}\,(\sigma) \leqslant \dim(\mathcal{H})^2$.

(3) Conclude that the Holevo quantity $\chi(T)$ is attained as stated above.