EXERCISES 11

Excercise 1 (Channels with vanishing classical capacity)**.** Let $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ denote a quantum channel. Show that $C(T) = 0$ if and only if T is constant, i.e., there exists a quantum state $\sigma \in D(\mathcal{H}_B)$ such that $T(X) = \operatorname{Tr}[X]\,\sigma$.

Excercise 2 (Entropy optimization)**.** For $k \in \{1, \ldots, N\}$ and $\delta \leqslant \frac{1}{k}$ let $\mathcal{P}_{\delta,k}(N)$ denote the set of all probability distributions $p \in \mathcal{P}(\{1, \ldots, N\})$ such that $p(1), \cdots, p(k) \geqslant \delta$. Show that

$$\max_{p \in \mathcal{P}_\delta(N)} = -k\delta \log(\delta) - (1 - \delta k) \log\left(\frac{1 - \delta k}{N - k}\right),$$

and that the optimum is achieved for the probability distribution with

$$p(1) = \cdots = p(k) = \delta \quad \text{and} \quad p(k+1) = \cdots = p(N) = \frac{1 - \delta k}{N - k}.$$

Hint: Use Lagrange multipliers.

Excercise 3 (Holevo information and minimal output entropy)**.** Consider a quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ and define the minimal output entropy

$$H_{\min}(T) = \inf\left(H(T(\rho)) \;:\; \rho \in D(\mathcal{H}_A)\right).$$

(1) Argue that the infimum in the definition of $H_{\min}$ is attained.

(2) Let $N = \dim(\mathcal{H}_B)^2$ and consider an orthogonal basis

$$\{U_n \;:\; n \in \{1, \ldots, N\}\} \subset \mathcal{U}(\mathcal{H}_B),$$

of the Hilbert-Schmidt inner product space $B(\mathcal{H}_B)$ consisting of unitary operators. Define a quantum channel $\tilde{T} : B(\mathbb{C}^N \otimes \mathcal{H}_A) \to B(\mathcal{H}_B)$ by

$$\tilde{T}(X) = \sum_{n=1}^{N} \operatorname{Ad}_{U_n} \circ T\left((\langle n| \otimes \mathbb{1}_{\mathcal{H}_A}) X (|n\rangle \otimes \mathbb{1}_{\mathcal{H}_A})\right),$$

and show that

$$\chi\left(\tilde{T}\right) = \log\left(\dim(\mathcal{H}_B)\right) - H_{\min}(T).$$

Hint: Consider the ensemble

$$\{\frac{1}{N}, |n\rangle\langle n| \otimes \rho\},$$

where $\rho \in D(\mathcal{H})$ satisfies $H(T(\rho)) = H_{\min}(T)$.

Excercise 4. For a quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ with Stinespring dilation

$$T(X) = \operatorname{Tr}_E\left[VXV^\dagger\right],$$

a complementary channel $T^c : B(\mathcal{H}_A) \to B(\mathcal{H}_E)$ (as discussed in Lecture 4) is given by

$$T^c(X) = \operatorname{Tr}_B\left[VXV^\dagger\right].$$

The goal of this exercise is to construct another family of quantum channels with additive Holevo information using complementary channels.

(1) Show that

$$H_{\min}(T) = H_{\min}(T^c).$$

(2) Use the results from the Exercise 3, to show that

$$H_{\min}(T \otimes S) = H_{\min}(T) + H_{\min}(S),$$

for any $T \in \mathrm{EB}(\mathcal{H}_A \to \mathcal{H}_B)$ and any quantum channel $S : B(\mathcal{H}_C) \to B(\mathcal{H}_D)$. Can you deduce a similar equality for T^c and S^c?

(3) Show that

$$\chi\left(\tilde{T^c} \otimes S\right) = \chi\left(\tilde{T^c}\right) + \chi(S),$$

for any quantum channel $S : B(\mathcal{H}_C) \to B(\mathcal{H}_D)$ whenever $T \in \mathrm{EB}(\mathcal{H}_A \to \mathcal{H}_B)$. Conclude a formula for the classical capacity $C(\tilde{T^c})$ for any $T \in \mathrm{EB}(\mathcal{H}_A \to \mathcal{H}_B)$.

(4) Compute a complementary channel T^c of an entanglement breaking channel T. Start with a Kraus decomposition of T with Kraus operators of rank 1 (Why does such a decomposition exist?).

(5) Consider a Schur multiplier $S_A : B(\mathbb{C}^d) \to B(\mathbb{C}^d)$ given by $S_A(X) = A \odot X$ for any $X \in B(\mathbb{C}^d)$, where $\odot$ denotes the entrywise product in the computational basis. Show that $S_A = T^c$ for some entanglement breaking channel T.