EXERCISES 13

Excercise 1 (The erasure channel)**.** The *qubit erasure channel* $E_\lambda : B(\mathbb{C}^2) \to B(\mathbb{C}^3)$ with erasure probability $\lambda \in [0,1]$ is given by

$$E_\lambda(X) = (1-\lambda)X \oplus 0 + \lambda \operatorname{Tr}[X] \, |3\rangle\langle 3|,$$

where for $X = (X_{ij})_{i,j}$ we have

$$X \oplus 0 = \begin{pmatrix} X_{11} & X_{12} & 0 \\ X_{21} & X_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$|3\rangle\langle 3| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(1) Compute a Stinespring dilation and a complementary channel of E_λ.
(2) Show that E_λ is antidegradable, whenever $\lambda \geqslant 1/2$.
(3) Is $\vartheta_3 \circ E_{1/2}$ completely positive?

Excercise 2 (A fidelity lemma)**.** We want to show that

$$1 + F(\rho,\sigma) = \max_{\tau \in D(\mathcal{H}_A)} \left(F(\rho,\tau)^2 + F(\sigma,\tau)^2 \right),$$

for all quantum states $\rho, \sigma \in D(\mathcal{H}_A)$. For this, we follow the steps below:

(1) Let $\mathcal{H}$ denote a complex Euclidean space. Show that

$$\| |v\rangle\langle v| + |w\rangle\langle w| \|_\infty = 1 + |\langle v|w\rangle|,$$

for any pair of normalized vectors $|v\rangle, |w\rangle \in \mathcal{H}$.

(2) Now, find purifications $|\psi_{EA}\rangle, |\phi_{EA}\rangle \in \mathcal{H}_E \otimes \mathcal{H}_A$ of ρ and σ, respectively, such that

$$F(\rho,\sigma) = |\langle \psi_{EA} | \phi_{EA} \rangle|,$$

and choose

$$\tau = \operatorname{Tr}_E \left[|\gamma_{EA}\rangle\langle\gamma_{EA}| \right],$$

where $|\gamma_{EA}\rangle \in \mathcal{H}_E \otimes \mathcal{H}_A$ is the normalized vector satisfying

$$\langle \gamma_{EA}| \left(|\psi_{EA}\rangle\langle\psi_{EA}| + |\phi_{EA}\rangle\langle\phi_{EA}| \right) |\gamma_{EA}\rangle = \| |\psi_{EA}\rangle\langle\psi_{EA}| + |\phi_{EA}\rangle\langle\phi_{EA}| \|_\infty.$$

Show that τ satisfies

$$1 + F(\rho,\sigma) \leqslant F(\rho,\tau)^2 + F(\sigma,\tau)^2.$$

Hint: Remember Uhlmann's theorem.

(3) Consider now an arbitrary quantum state $\tau \in D(\mathcal{H}_A)$. Show that there are purifications $|\psi_{EA}\rangle, |\phi_{EA}\rangle, |\gamma_{EA}\rangle \in \mathcal{H}_E \otimes \mathcal{H}_A$ of ρ, σ and τ, respectively, such that

$$|\langle \psi_{EA} | \gamma_{EA} \rangle| = F(\rho,\tau)$$
$$|\langle \phi_{EA} | \gamma_{EA} \rangle| = F(\sigma,\tau).$$

Use those to show that

$$1 + F(\rho,\sigma) \geqslant F(\rho,\tau)^2 + F(\sigma,\tau)^2.$$

Excercise 3 (Technical lemma)**.** Let $\mathcal{H}, \mathcal{H}'$ denote complex Euclidean spaces and consider a quantum channel $T : B(\mathcal{H}') \to B(\mathcal{H})$ satisfying

$$F\left(\omega_{\mathcal{H}}, (\mathrm{id}_{B(\mathcal{H})} \otimes T)\left(|\phi\rangle\langle\phi|\right)\right) \geqslant 1 - \delta,$$

for some pure state $|\phi\rangle \in \mathcal{H} \otimes \mathcal{H}'$. Show that there exists a number $r \leqslant \dim(\mathcal{H})$ and a pure state $|\Omega_r\rangle \in \mathcal{H} \otimes \mathcal{H}$ of the form

$$|\Omega_r\rangle = \frac{1}{\sqrt{r}} \sum_{i=1}^{r} \overline{|a_i\rangle} \otimes |a_i\rangle$$

for some orthonormal set $\{|a_i\rangle\}_{i=1}^r \subset \mathcal{H}$, and a quantum channel $E : B(\mathcal{H}) \to B(\mathcal{H}')$ with

$$F\left(\omega_{\mathcal{H}}, (\mathrm{id}_{B(\mathcal{H})} \otimes T \circ E)\left(|\Omega_r\rangle\langle\Omega_r|\right)\right) \geqslant 1 - 4\delta.$$

You may follow the steps below:

(1) Consider the Schmidt decomposition

$$|\phi\rangle = \sum_{i=1}^{r} \lambda_i \overline{|a_i\rangle} \otimes |b_i\rangle,$$

and define

$$|\tilde{\phi}\rangle = \frac{1}{\sqrt{r}} \sum_{i=1}^{r} \overline{|a_i\rangle} \otimes |b_i\rangle.$$

Using the data-processing inequality of the fidelity for the partial trace, show that

$$F\left(|\tilde{\phi}\rangle\langle\tilde{\phi}|, |\phi\rangle\langle\phi|\right)^2 \geqslant 1 - \delta.$$

(2) Use the fidelity lemma from exercise 2 on this sheet to show that

$$1 + F\left(\omega_{\mathcal{H}'}, (\mathrm{id}_{B(\mathcal{H}')} \otimes T)\left(|\tilde{\phi}\rangle\langle\tilde{\phi}|\right)\right) \geqslant 2(1-\delta)^2.$$

(3) Find a quantum channel E such that

$$(\mathrm{id}_{\mathcal{H}} \otimes E)\left(|\Omega_r\rangle\langle\Omega_r|\right) = |\tilde{\phi}\rangle\langle\tilde{\phi}|,$$

and conclude the desired lemma.

Excercise 4 (Decoupling implies code)**.** Consider a quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ with Stinespring isometry $V : \mathcal{H}_A \to \mathcal{H}_E \otimes \mathcal{H}_B$ and a pure state $|\phi_{RA}\rangle \in \mathcal{H}_R \otimes \mathcal{H}_A$ for some complex Euclidean space $\mathcal{H}_R$. For the pure state

$$|\psi^{REB}\rangle = (\mathbb{1}_R \otimes V)|\phi_{RA}\rangle \in \mathcal{H}_R \otimes \mathcal{H}_E \otimes \mathcal{H}_B$$

we assume that

$$\| \mathrm{Tr}_B\left[|\psi^{REB}\rangle\langle\psi^{REB}|\right] - \frac{\mathbb{1}_R}{\dim(\mathcal{H}_R)} \otimes \tau^E \|_1 \leqslant \epsilon,$$

for some $\tau^E \in D(\mathcal{H}_E)$. Show that under these assumptions there exists a quantum channel $D : B(\mathcal{H}_B) \to B(\mathcal{H}_R)$ satisfying

$$F(\omega_R, (\mathrm{id}_R \otimes D \circ T)\left(|\phi^{RA}\rangle\langle\phi^{RA}|\right) \geqslant 1 - \epsilon.$$

Hint: Consider two purifications of

$$\frac{\mathbb{1}_R}{\dim(\mathcal{H}_R)} \otimes \tau^E.$$

The first one by applying Uhlmann's theorem for the fidelity with the quantum state $\mathrm{Tr}_B\left[|\psi^{REB}\rangle\langle\psi^{REB}|\right]$ and the second one by purifying the two tensor factors separately. Then, use a previous exercise to relate the two purifying systems by a quantum channel to construct the quantum channel D.