EXERCISES 13

Excercise 1 (The erasure channel)**.** The *qubit erasure channel* $E_\lambda : B(\mathbb{C}^2) \to B(\mathbb{C}^3)$ with erasure probability $\lambda \in [0,1]$ is given by

$$E_\lambda(X) = (1-\lambda)X \oplus 0 + \lambda \operatorname{Tr}[X] |3\rangle\langle 3|,$$

where for $X = (X_{ij})_{i,j}$ we have

$$X \oplus 0 = \begin{pmatrix} X_{11} & X_{12} & 0 \\ X_{21} & X_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$|3\rangle\langle 3| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(1) Compute a Stinespring dilation and a complementary channel of E_λ.
(2) Show that E_λ is antidegradable, whenever $\lambda \geqslant 1/2$.
(3) Is $\vartheta_3 \circ E_{1/2}$ completely positive?

Proof. (1) Consider the isometry $V : \mathbb{C}^2 \to \mathbb{C}^3$ given by

$$W_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and note that

$$W_1 X W_1^\dagger = X \oplus 0 \in B(\mathbb{C}^3),$$

for any $X \in B(\mathbb{C}^2)$. Moreover, consider the operators

$$W_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad W_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and note that

$$W_2 X W_2^\dagger + W_3 X W_3^\dagger = \operatorname{Tr}[X] |3\rangle\langle 3|.$$

We find that

$$E_\lambda(X) = (1-\lambda) W_1 X W_1^\dagger + \lambda W_2 X W_2^\dagger + \lambda W_3 X W_3^\dagger,$$

and therefore we have a Kraus decomposition with Kraus operators

$$K_1 = \sqrt{1-\lambda} W_1$$
$$K_1 = \sqrt{\lambda} W_2$$
$$K_1 = \sqrt{\lambda} W_3.$$

Now, a Stinespring isometry for E_λ can be constructed as

$$V = \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} \in B(\mathbb{C}^2, \mathbb{C}^3 \otimes \mathbb{C}^3).$$

We can verify that

$$VXV^\dagger = \begin{pmatrix} K_1XK_1^\dagger & K_1XK_2^\dagger & K_1XK_3^\dagger \\ K_2XK_1^\dagger & K_2XK_2^\dagger & K_2XK_3^\dagger \\ K_3XK_1^\dagger & K_3XK_2^\dagger & K_3XK_3^\dagger \end{pmatrix},$$

and a complementary quantum channel is given by

$$E_\lambda^c(X) = \begin{pmatrix} \operatorname{Tr}\left[K_1XK_1^\dagger\right] & \operatorname{Tr}\left[K_1XK_2^\dagger\right] & \operatorname{Tr}\left[K_1XK_3^\dagger\right] \\ \operatorname{Tr}\left[K_2XK_1^\dagger\right] & \operatorname{Tr}\left[K_2XK_2^\dagger\right] & \operatorname{Tr}\left[K_2XK_3^\dagger\right] \\ \operatorname{Tr}\left[K_3XK_1^\dagger\right] & \operatorname{Tr}\left[K_3XK_2^\dagger\right] & \operatorname{Tr}\left[K_3XK_3^\dagger\right] \end{pmatrix}$$

Using the particular form of the operators K_1, K_2 and K_3 it is easy to verify that

$$E_\lambda^c(X) = \begin{pmatrix} (1-\lambda)\operatorname{Tr}[X] & 0 & 0 \\ 0 & \lambda X_{11} & \lambda X_{12} \\ 0 & \lambda X_{21} & \lambda X_{22} \end{pmatrix}.$$

(2) Consider the unitary $U : \mathbb{C}^3 \to \mathbb{C}^3$ permuting the computational basis vectors as

$$U|1\rangle = |3\rangle, \quad U|2\rangle = |1\rangle, \quad U|3\rangle = |2\rangle,$$

and note that

$$\operatorname{Ad}_U \circ E_\lambda^c(X) = \begin{pmatrix} \lambda X_{11} & \lambda X_{12} & 0 \\ \lambda X_{21} & \lambda X_{22} & 0 \\ 0 & 0 & (1-\lambda)\operatorname{Tr}[X] \end{pmatrix} = E_{(1-\lambda)}(X).$$

Next, consider a quantum channel $D_\mu : B(\mathbb{C}^3) \to B(\mathbb{C}^3)$ given by

$$D_\mu(Y) = \mu Y + (1-\mu)\operatorname{Tr}[Y]\,|3\rangle\langle 3|,$$

and note that

$$D_\mu \circ \operatorname{Ad}_U \circ E_\lambda^c(X) = E_{1-\lambda\mu}(X).$$

For $\lambda \geqslant 1/2$ it is easy to see that $\lambda = 1 - \lambda\mu$ for

$$\mu = \frac{1}{\lambda} - 1 \in [0,1].$$

We conclude that E_λ for $\lambda \geqslant 1/2$ is antidegradable.

(3) The positive map $\vartheta_3 \circ E_{1/2}$ is not completely positive, since projecting its image to $\operatorname{span}\{|1\rangle, |2\rangle\}$ is a non-zero multiple of the transpose map.

□

Excercise 2 (A fidelity lemma)**.** We want to show that

$$1 + F(\rho,\sigma) = \max_{\tau \in D(\mathcal{H}_A)} \left(F(\rho,\tau)^2 + F(\sigma,\tau)^2\right),$$

for all quantum states $\rho, \sigma \in D(\mathcal{H}_A)$. For this, we follow the steps below:

(1) Let $\mathcal{H}$ denote a complex Euclidean space. Show that

$$\| |v\rangle\langle v| + |w\rangle\langle w| \|_\infty = 1 + |\langle v|w\rangle|,$$

for any pair of normalized vectors $|v\rangle, |w\rangle \in \mathcal{H}$.

(2) Now, find purifications $|\psi_{EA}\rangle, |\phi_{EA}\rangle \in \mathcal{H}_E \otimes \mathcal{H}_A$ of ρ and σ, respectively, such that

$$F(\rho,\sigma) = |\langle \psi_{EA}|\phi_{EA}\rangle|,$$

and choose

$$\tau = \operatorname{Tr}_E\left[|\gamma_{EA}\rangle\langle\gamma_{EA}|\right],$$

where $|\gamma_{EA}\rangle \in \mathcal{H}_E \otimes \mathcal{H}_A$ is the normalized vector satisfying

$$\langle\gamma_{EA}| \left(|\psi_{EA}\rangle\langle\psi_{EA}| + |\phi_{EA}\rangle\langle\phi_{EA}|\right) |\gamma_{EA}\rangle = \||\psi_{EA}\rangle\langle\psi_{EA}| + |\phi_{EA}\rangle\langle\phi_{EA}|\|_\infty.$$

Show that τ satisfies

$$1 + F(\rho,\sigma) \leqslant F(\rho,\tau)^2 + F(\sigma,\tau)^2.$$

Hint: Remember Uhlmann's theorem.

Theorem 0.1 (Uhlmann's theorem)**.** *Let $\mathcal{H}_A, \mathcal{H}_E$ denote complex Euclidean spaces and $\rho, \sigma \in D(\mathcal{H}_A)$ two quantum states. For any normalized vector $|\psi_{EA}\rangle \in \mathcal{H}_E \otimes \mathcal{H}_A$ satisfying $\mathrm{Tr}_E\left[|\psi_{EA}\rangle\langle\psi_{EA}|\right] = \rho$, we have*

$$F(\rho,\sigma) = \max\{|\langle\psi_{EA}|\phi_{EA}\rangle| \ : \ \mathrm{Tr}_E\left[|\phi_{EA}\rangle\langle\phi_{EA}|\right] = \sigma\}.$$

(3) Consider now an arbitrary quantum state $\tau \in D(\mathcal{H}_A)$. Show that there are purifications $|\psi_{EA}\rangle, |\phi_{EA}\rangle, |\gamma_{EA}\rangle \in \mathcal{H}_E \otimes \mathcal{H}_A$ of ρ, σ and τ, respectively, such that

$$|\langle\psi_{EA}|\gamma_{EA}\rangle| = F(\rho,\tau)$$
$$|\langle\phi_{EA}|\gamma_{EA}\rangle| = F(\sigma,\tau).$$

Use those to show that

$$1 + F(\rho,\sigma) \geqslant F(\rho,\tau)^2 + F(\sigma,\tau)^2.$$

Proof. (1) Consider a pair of normalized vectors $|v\rangle, |w\rangle \in \mathcal{H}$.

(2) By Uhlmann's theorem there exist purifications $|\psi_{EA}\rangle, |\phi_{EA}\rangle \in \mathcal{H}_E \otimes \mathcal{H}_A$ of ρ and σ, respectively, such that

$$F(\rho,\sigma) = |\langle\psi_{EA}|\phi_{EA}\rangle|.$$

We choose

$$\tau = \mathrm{Tr}_E\left[|\gamma_{EA}\rangle\langle\gamma_{EA}|\right],$$

where $|\gamma_{EA}\rangle \in \mathcal{H}_E \otimes \mathcal{H}_A$ is the normalized (eigen-)vector satisfying

$$\langle\gamma_{EA}| \left(|\psi_{EA}\rangle\langle\psi_{EA}| + |\phi_{EA}\rangle\langle\phi_{EA}|\right) |\gamma_{EA}\rangle = \||\psi_{EA}\rangle\langle\psi_{EA}| + |\phi_{EA}\rangle\langle\phi_{EA}|\|_\infty.$$

Now, we find that

$$\begin{aligned} 1 + F(\rho,\sigma) = 1 + |\langle\psi_{EA}|\phi_{EA}\rangle| &= \||\psi_{EA}\rangle\langle\psi_{EA}| + |\phi_{EA}\rangle\langle\phi_{EA}|\|_\infty \\ &= \langle\gamma_{EA}| \left(|\psi_{EA}\rangle\langle\psi_{EA}| + |\phi_{EA}\rangle\langle\phi_{EA}|\right) |\gamma_{EA}\rangle \\ &= |\langle\gamma_{EA}|\psi_{EA}\rangle|^2 + |\langle\gamma_{EA}|\phi_{EA}\rangle|^2 \\ &\leqslant F(\rho,\tau)^2 + F(\sigma,\tau)^2, \end{aligned}$$

where we used Uhlmann's theorem for the final inequality.

(3) Consider now an arbitrary quantum state $\tau \in D(\mathcal{H}_A)$. By Uhlmann's theorem there are purifications $|\psi_{EA}\rangle, |\phi_{EA}\rangle, |\gamma_{EA}\rangle \in \mathcal{H}_E \otimes \mathcal{H}_A$ of ρ, σ and τ, respectively, such that

$$|\langle\psi_{EA}|\gamma_{EA}\rangle| = F(\rho,\tau)$$
$$|\langle\phi_{EA}|\gamma_{EA}\rangle| = F(\sigma,\tau).$$

By Uhlmann's theorem we have

$$\begin{aligned} 1 + F(\rho,\sigma) \geqslant 1 + |\langle\psi_{EA}|\phi_{EA}\rangle| &= \||\psi_{EA}\rangle\langle\psi_{EA}| + |\phi_{EA}\rangle\langle\phi_{EA}|\|_\infty \\ &\geqslant \langle\gamma_{EA}| \left(|\psi_{EA}\rangle\langle\psi_{EA}| + |\phi_{EA}\rangle\langle\phi_{EA}|\right) |\gamma_{EA}\rangle \\ &= F(\rho,\tau)^2 + F(\sigma,\tau)^2. \end{aligned}$$

This finishes the proof.

□

Excercise 3 (Technical lemma)**.** Let $\mathcal{H}, \mathcal{H}'$ denote complex Euclidean spaces and consider a quantum channel $T : B(\mathcal{H}') \to B(\mathcal{H})$ satisfying

$$F\left(\omega_{\mathcal{H}}, (\mathrm{id}_{B(\mathcal{H})} \otimes T)(|\phi\rangle\langle\phi|)\right) \geqslant 1 - \delta,$$

for some pure state $|\phi\rangle \in \mathcal{H} \otimes \mathcal{H}'$. Show that there exists a number $r \leqslant \dim(\mathcal{H})$ and a pure state $|\Omega_r\rangle \in \mathcal{H} \otimes \mathcal{H}$ of the form

$$|\Omega_r\rangle = \frac{1}{\sqrt{r}} \sum_{i=1}^{r} \overline{|a_i\rangle} \otimes |a_i\rangle$$

for some orthonormal set $\{|a_i\rangle\}_{i=1}^r \subset \mathcal{H}$, and a quantum channel $E : B(\mathcal{H}) \to B(\mathcal{H}')$ with

$$F\left(\omega_{\mathcal{H}}, (\mathrm{id}_{B(\mathcal{H})} \otimes T \circ E)(|\Omega_r\rangle\langle\Omega_r|)\right) \geqslant 1 - 4\delta.$$

You may follow the steps below:

(1) Consider the Schmidt decomposition

$$|\phi\rangle = \sum_{i=1}^{r} \lambda_i \overline{|a_i\rangle} \otimes |b_i\rangle,$$

and define

$$|\tilde{\phi}\rangle = \frac{1}{\sqrt{r}} \sum_{i=1}^{r} \overline{|a_i\rangle} \otimes |b_i\rangle.$$

Using the data-processing inequality of the fidelity for the partial trace, show that

$$F\left(|\tilde{\phi}\rangle\langle\tilde{\phi}|, |\phi\rangle\langle\phi|\right)^2 \geqslant 1 - \delta.$$

(2) Use the fidelity lemma from exercise 2 on this sheet to show that

$$1 + F\left(\omega_{\mathcal{H}'}, (\mathrm{id}_{B(\mathcal{H}')} \otimes T)\left(|\tilde{\phi}\rangle\langle\tilde{\phi}|\right)\right) \geqslant 2(1-\delta)^2.$$

(3) Find a quantum channel E such that

$$(\mathrm{id}_{\mathcal{H}} \otimes E)(|\Omega_r\rangle\langle\Omega_r|) = |\tilde{\phi}\rangle\langle\tilde{\phi}|,$$

and conclude the desired lemma.

Proof. Consider a pure quantum state $|\phi\rangle \in \mathcal{H}' \otimes \mathcal{H}$ satisfying

$$F\left(\omega_{\mathcal{H}'}, (\mathrm{id}_{B(\mathcal{H}')} \otimes T)(|\phi\rangle\langle\phi|)\right) \geqslant 1 - \delta,$$

and consider the Schmidt decomposition

$$|\phi\rangle = \sum_{i=1}^{r} \lambda_i \overline{|a_i\rangle} \otimes |b_i\rangle,$$

with $\lambda_i \geqslant 0$ satisfying $\sum_{i=1}^r \lambda_i^2 = 1$ and orthonormal sets $\{|a_i\rangle\}_{i=1}^r \subset \mathcal{H}'$ and $\{|b_i\rangle\}_{i=1}^r \subset \mathcal{H}$. Next, consider the pure state

$$|\tilde{\phi}\rangle = \frac{1}{\sqrt{r}} \sum_{i=1}^{r} \overline{|a_i\rangle} \otimes |b_i\rangle.$$

(1) We have

$$F\left(|\tilde{\phi}\rangle\langle\tilde{\phi}|, |\phi\rangle\langle\phi|\right)^2 = |\langle\tilde{\phi}|\phi\rangle|^2 = \left(\sum_{i=1}^{r} \frac{\lambda_i}{\sqrt{r}}\right)^2 \geqslant 1 - \delta,$$

since (by using the data-processing inequality) we have

$$\left(\sum_{i=1}^{r} \frac{\lambda_i}{\sqrt{r}}\right)^2 \geqslant \left(\sum_{i=1}^{r} \frac{\lambda_i}{\sqrt{d}}\right)^2 = F(\frac{1}{d}\mathbb{1}_{\mathcal{H}}, \mathrm{Tr}_{B(\mathcal{H})}[|\phi\rangle\langle\phi|]) \geqslant F\left(\omega_{\mathcal{H}'}, (\mathrm{id}_{B(\mathcal{H}')} \otimes T)(|\phi\rangle\langle\phi|)\right) \geqslant 1-\delta.$$

(2) By Exercise 2 and the data-processing inequality for the fidelity, we have

$$\begin{aligned}
&1 + F\left(\omega_{\mathcal{H}'}, (\mathrm{id}_{B(\mathcal{H}')} \otimes T)\left(|\tilde{\phi}\rangle\langle\tilde{\phi}|\right)\right) \\
&\geqslant F\left(\omega_{\mathcal{H}'}, (\mathrm{id}_{B(\mathcal{H}')} \otimes T)\left(|\phi\rangle\langle\phi|\right)\right)^2 + F\left((\mathrm{id}_{B(\mathcal{H}')} \otimes T)\left(|\tilde{\phi}\rangle\langle\tilde{\phi}|\right), (\mathrm{id}_{B(\mathcal{H}')} \otimes T)\left(|\phi\rangle\langle\phi|\right)\right)^2 \\
&\geqslant F\left(\omega_{\mathcal{H}'}, (\mathrm{id}_{B(\mathcal{H}')} \otimes T)\left(|\phi\rangle\langle\phi|\right)\right)^2 + F\left(|\tilde{\phi}\rangle\langle\tilde{\phi}|, |\phi\rangle\langle\phi|\right)^2 \\
&\geqslant 2(1-\delta)^2,
\end{aligned}$$

We conclude that

$$F\left(\omega_{\mathcal{H}'}, (\mathrm{id}_{B(\mathcal{H}')} \otimes T)\left(|\tilde{\phi}\rangle\langle\tilde{\phi}|\right)\right) \geqslant 1 - 4\delta.$$

(3) Finally, consider the operator

$$V = \sum_{i=1}^{r} |b_i\rangle\langle a_i| \in B(\mathcal{H}, \mathcal{H}')$$

and define $E : B(\mathcal{H}) \to B(\mathcal{H}')$ by

$$E(X) = VXV^\dagger + \mathrm{Tr}\left[(\mathbb{1}_{\mathcal{H}} - V^\dagger V)X\right].$$

Then, we have

$$(\mathrm{id}_{\mathcal{H}} \otimes E)\left(|\Omega_r\rangle\langle\Omega_r|\right) = |\tilde{\phi}\rangle\langle\tilde{\phi}|,$$

and the statement of the lemma follows.

□

Excercise 4 (Decoupling implies code)**.** Consider a quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ with Stinespring isometry $V : \mathcal{H}_A \to \mathcal{H}_E \otimes \mathcal{H}_B$ and a pure state $|\phi_{RA}\rangle \in \mathcal{H}_R \otimes \mathcal{H}_A$ for some complex Euclidean space $\mathcal{H}_R$. For the pure state

$$|\psi^{REB}\rangle = (\mathbb{1}_R \otimes V)|\phi_{RA}\rangle \in \mathcal{H}_R \otimes \mathcal{H}_E \otimes \mathcal{H}_B$$

we assume that

$$\| \mathrm{Tr}_B\left[|\psi^{REB}\rangle\langle\psi^{REB}|\right] - \frac{\mathbb{1}_R}{\dim(\mathcal{H}_R)} \otimes \tau^E \|_1 \leqslant \epsilon,$$

for some $\tau^E \in D(\mathcal{H}_E)$. Show that under these assumptions there exists a quantum channel $D : B(\mathcal{H}_B) \to B(\mathcal{H}_R)$ satisfying

$$F(\omega_R, (\mathrm{id}_R \otimes D \circ T)\left(|\phi^{RA}\rangle\langle\phi^{RA}|\right) \geqslant 1 - \frac{1}{2}\epsilon.$$

Hint: Consider two purifications of

$$\frac{\mathbb{1}_R}{\dim(\mathcal{H}_R)} \otimes \tau^E.$$

The first one by applying Uhlmann's theorem for the fidelity with the quantum state $\mathrm{Tr}_B\left[|\psi^{REB}\rangle\langle\psi^{REB}|\right]$ and the second one by purifying the two tensor factors separately. Then, use a previous exercise to relate the two purifying systems by a quantum channel to construct the quantum channel D.

Proof. As in the hint, we consider two purifications of the quantum state

$$\frac{\mathbb{1}_R}{\dim(\mathcal{H}_R)} \otimes \tau^E \in D\left(\mathcal{H}_R \otimes \mathcal{H}_E\right).$$

The first purification $|\alpha^{REB}\rangle \in \mathcal{H}_R \otimes \mathcal{H}_E \otimes \mathcal{H}_B$ satisfies

$$F\left(\mathrm{Tr}_B\left[|\psi^{REB}\rangle\langle\psi^{REB}|\right], \frac{\mathbb{1}_R}{\dim(\mathcal{H}_R)} \otimes \tau^E\right) = |\langle\psi^{REB}|\alpha^{REB}\rangle|,$$

and its existence follows from Uhlmann's theorem. The second purification $|\beta^{RR'EE'}\rangle \in \mathcal{H}_R \otimes \mathcal{H}_R \otimes \mathcal{H}_E \otimes \mathcal{H}_E$ is given by

$$|\beta^{RR'EE'}\rangle = |\Omega_R\rangle \otimes |\phi^{EE'}\rangle,$$

where $|\Omega_R\rangle \in \mathcal{H}_R \otimes \mathcal{H}_R$ denotes the (normalized) maximally entangled state, and $|\phi^{EE'}\rangle \in \mathcal{H}_E \otimes \mathcal{H}_E$ some purification of τ^E. By construction, these purifications satisfy

$$\frac{\mathbb{1}_R}{\dim(\mathcal{H}_R)} \otimes \tau^E = \mathrm{Tr}_B \left[|\alpha^{REB}\rangle\langle\alpha^{REB}|\right] = \mathrm{Tr}_{R'E'} \left[|\beta^{RR'EE'}\rangle\langle\beta^{RR'EE'}|\right],$$

and by exercise 3 on sheet 7, we find a quantum channel $S : B(\mathcal{H}_B) \to B(\mathcal{H}_R \otimes \mathcal{H}_E)$ such that

$$(\mathrm{id}_{RE} \otimes S)\left(|\alpha^{REB}\rangle\langle\alpha^{REB}|\right) = |\beta^{RR'EE'}\rangle\langle\beta^{RR'EE'}|.$$

Setting $D = \mathrm{Tr}_E \circ S$, we can verify that

$$\begin{aligned}
&F(\omega_R, (\mathrm{id}_R \otimes D \circ T)\left(|\phi^{RA}\rangle\langle\phi^{RA}|\right) \\
&\geqslant F(|\beta^{RR'EE'}\rangle\langle\beta^{RR'EE'}|, (\mathrm{id}_{RE} \otimes S)\left(|\psi^{REB}\rangle\langle\psi^{REB}|\right) \\
&= F((\mathrm{id}_{RE} \otimes S)\left(|\alpha^{REB}\rangle\langle\alpha^{REB}|\right), (\mathrm{id}_{RE} \otimes S)\left(|\psi^{REB}\rangle\langle\psi^{REB}|\right) \\
&\geqslant F\left(|\alpha^{REB}\rangle\langle\alpha^{REB}|, |\psi^{REB}\rangle\langle\psi^{REB}|\right),
\end{aligned}$$

where we used the monotonicity of the fidelity under partial traces in the first inequality, and the monotonicity of the fidelity under quantum channels in the second inequality. We continue as follows:

$$\begin{aligned}
F\left(|\alpha^{REB}\rangle\langle\alpha^{REB}|, |\psi^{REB}\rangle\langle\psi^{REB}|\right) &= |\langle\alpha^{REB}|\psi^{REB}\rangle| \\
&= F\left(\mathrm{Tr}_B\left[|\psi^{REB}\rangle\langle\psi^{REB}|\right], \frac{\mathbb{1}_R}{\dim(\mathcal{H}_R)} \otimes \tau^E\right) \\
&\geqslant 1 - \frac{1}{2}\| \mathrm{Tr}_B\left[|\psi^{REB}\rangle\langle\psi^{REB}|\right] - \frac{\mathbb{1}_R}{\dim(\mathcal{H}_R)} \otimes \tau^E\|_1 \\
&\geqslant 1 - \frac{1}{2}\epsilon.
\end{aligned}$$

We conclude that

$$F(\omega_R, (\mathrm{id}_R \otimes D \circ T)\left(|\phi^{RA}\rangle\langle\phi^{RA}|\right) \geqslant 1 - \frac{1}{2}\epsilon,$$

which finishes the proof. □