EXERCISES 2

Excercise 1 (Sums of projections)**.** For a complex Euclidean space $\mathcal{H}$ consider projections $\{P_n\}_{n=1}^N \subset B(\mathcal{H})$ such that $\sum_{n=1}^N P_n = \mathbb{1}_{\mathcal{H}}$. Show that $P_i P_j = 0$ all $i \neq j$.

Excercise 2 (The Deutsch algorithm)**.** Consider a function $f : \{0,1\} \to \{0,1\}$ and define a linear operator $U_f \in B(\mathbb{C}^4)$ acting as

$$U_f(|x\rangle \otimes |y\rangle) = |x\rangle \otimes |y \oplus f(x)\rangle,$$

on the computational basis[1] for $x, y \in \{0,1\}$, and where $\oplus$ denotes the binary addition. Furthermore, consider the unitary operator $H \in \mathcal{U}\left(\mathbb{C}^2\right)$ given by

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

(1) Show that U_f is a unitary.
(2) Define a unitary $C_f \in \mathcal{U}\left(\mathbb{C}^4\right)$ by

$$C_f = (H \otimes \mathbb{1}_{\mathbb{C}^4}) \circ U_f \circ (H \otimes H),$$

and compute the state

$$|\psi_f\rangle = C_f\left(|0\rangle \otimes |1\rangle\right).$$

(3) When you measure the first qubit using the PVM $\{|0\rangle\langle 0|, |1\rangle\langle 1|\} \subset \mathrm{Proj}\left(\mathbb{C}^2\right)$, what do you learn about the function f? Why is this surprising?

Excercise 3 (The qubit)**.** How can we visualize quantum states on the complex Euclidean space $\mathcal{H} = \mathbb{C}^2$?

(1) Consider the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that $\{\mathbb{1}_2, \sigma_1, \sigma_2, \sigma_3\}$ is an orthogonal basis of the Hilbert-Schmidt inner product space $B(\mathcal{H})$ and that their real span is $B(\mathcal{H})_{sa}$.
(2) Show that we have

$$\left(\sum_{i=1}^3 x_i \sigma_i\right)^* \left(\sum_{i=1}^3 x_i \sigma_i\right) = \|x\|_2^2 \mathbb{1}_2,$$

for any $x \in \mathbb{R}^3$. Conclude that the matrix

$$\rho_x = \frac{1}{2}\mathbb{1}_2 + \frac{1}{2}\sum_{i=1}^3 x_i \sigma_i,$$

is a quantum state, i.e., positive and of unit trace, for any vector $x \in \mathbb{R}^3$ satisfying $\|x\|_2 \leqslant 1$.
(3) Show that every quantum state $\rho \in D(\mathbb{C}^2)$ can be written as $\rho = \rho_x$ for some $x \in \mathbb{R}^3$. Therefore, we can identify $D(\mathbb{C}^2)$ with the unit ball in the 2-norm. What are the pure states in this picture? What quantum state is at the center of the Bloch ball?
(4) Argue that $D(\mathbb{C}^3)$ is not isomorphic to the unit ball for any norm.

[1] which we index starting from 0 in this exercise.

Excercise 4 (Ensemble representation theorem)**.** For a complex Euclidean space of dimension $\dim(\mathcal{H}) = d$ consider two sets $\{|\psi_n\rangle\}_{n=1}^N \subset \mathcal{H}$ and $\{|\phi_k\rangle\}_{k=1}^K \subset \mathcal{H}$ of vectors. We aim to show that

$$\sum_{n=1}^{N} |\psi_n\rangle\langle\psi_n| = \sum_{k=1}^{K} |\phi_k\rangle\langle\phi_k|.$$

if and only if there exists a unitary operator $U \in \mathcal{U}\left(\mathbb{C}^N\right)$ such that

$$|\psi_n\rangle = \sum_{k=1}^{K} U_{nk}|\phi_k\rangle,$$

where the smaller set of vectors is extended by zero vectors. To show this result apply the following steps, or find your own proof:

(1) Show the easy direction.
(2) Consider a set of orthonormal vectors $\{|e_k\rangle\}_{k=1}^K \subset \mathcal{H}$ and let $\{|\psi_n\rangle\}_{n=1}^N \subset \mathcal{H}$ for $N \geqslant K$ satisfy

$$\sum_{n=1}^{N} |\psi_n\rangle\langle\psi_n| = \sum_{k=1}^{K} |e_k\rangle\langle e_k|.$$

Show that

$$\{\left(|e_k\rangle\right\}_{k=1}^K)^{\perp} \subseteq \left(\{|\psi_n\rangle\}_{n=1}^N\right)^{\perp},$$

where $\mathcal{S}^{\perp}$ denotes the orthogonal complement of a set $\mathcal{S} \subseteq \mathcal{H}$. Then, construct a unitary $U \in \mathcal{U}\left(\mathbb{C}^N\right)$ such that

$$\sum_{n=1}^{N} U_{kn}|\psi_n\rangle = |e_k\rangle,$$

and

$$\sum_{k=1}^{K} \overline{U_{kn}}|e_k\rangle = |\psi_n\rangle,$$

for each $n \in \{1, \ldots, N\}$ and $k \in \{1, \ldots, K\}$.
(3) Consider next the case, where $\{|e_k\rangle\}_{k=1}^K \subset \mathcal{H}$ are the eigenvectors corresponding to non-zero eigenvalues $\{\lambda_1, \ldots, \lambda_K\}$ of the operator

$$A = \sum_{n=1}^{N} |\psi_n\rangle\langle\psi_n|.$$

Construct a unitary operator $U \in \mathcal{U}\left(\mathbb{C}^N\right)$ such that

$$\sum_{n=1}^{N} U_{kn}|\psi_n\rangle = \sqrt{\lambda_k}|e_k\rangle,$$

and

$$\sum_{k=1}^{K} \overline{U_{kn}}\sqrt{\lambda_k}|e_k\rangle = |\psi_n\rangle,$$

for each $n \in \{1, \ldots, N\}$ and $k \in \{1, \ldots, K\}$.
(4) Finally, combine the insights of the previous points to prove the theorem stated in the beginning of this exercise.

Excercise 5 (Bonus question)**.** A symmetrical informationally-complete PVM (also known as a SIC POVM) on $\mathbb{C}^d$ is a set of d^2 rank-1 projections

$$\{|\psi_1\rangle\langle\psi_1|, \ldots, |\psi_{d^2}\rangle\langle\psi_{d^2}|\} \subset \mathrm{Proj}\left(\mathbb{C}^d\right)$$

such that

$$|\langle\psi_i|\psi_j\rangle|^2 = \frac{d\delta_{ij}+1}{d+1},$$

for any $i, j \in \{1, \ldots, d^2\}$. Construct a SIC POVM on $\mathbb{C}^2$.