EXERCISES 3

Excercise 1 (The Choi-Jamiolkowski isomorphism)**.**

(1) Prove the necklace identities for the operator-vector correspondence from the lecture notes.
(2) Consider a linear map $L : B(H_A) \to B(H_B)$ given by $L(X) = \mathrm{Tr}\left[A^\dagger X\right] B$ for $A \in B(\mathcal{H}_A)$ and $B \in B(\mathcal{H}_B)$. What is the Choi operator of L?
(3) Compute the Choi operator of the transpose map $\vartheta : B(\mathcal{H}) \to B(\mathcal{H})$ in the computational basis. Is this map completely positive?
(4) Consider linear maps $L_1 : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ and $L_2 : B(\mathcal{H}_C) \to B(\mathcal{H}_D)$ for Euclidean spaces $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C, \mathcal{H}_D$. What is the Choi matrix of the tensor product map $T : B(\mathcal{H}_A \otimes \mathcal{H}_C) \to B(\mathcal{H}_B \otimes \mathcal{H}_D)$ given by $T = S_1 \otimes S_2$?

Excercise 2 (Depolarizing channel)**.** For $p \in [0, 1]$ we define the *depolarizing channel* $D_p : B(\mathbb{C}^d) \to B(\mathbb{C}^d)$ by

$$D_p(X) = (1 - p)\,\mathrm{Tr}\,[X]\,\mathbb{1}_d + pX,$$

for any $X \in B(\mathbb{C}^d)$.

(1) Compute the Choi operator of D_p. Is D_p a quantum channel?
(2) Compute the Choi-Kraus representation of D_p.
(3) Compute the Stinespring dilation of D_p.
(4) Compute the complementary channel D_p^c for dimension $d = 2$.

Excercise 3 (Superdense coding)**.** Consider two researchers named Alice and Bob who each have a laboratory. Alice's laboratory contains a quantum system labelled 'A' and Bob's laboratory contains a quantum system labelled 'B'. Each of these quantum systems is a qubit, i.e., their state space H_A and H_B are equal to $\mathbb{C}^2$. Initially, 'A' and 'B' are in a (*normalized*!) maximally entangled pure state, i.e., their state is given by $\omega_{AB} = |\Omega_{AB}\rangle\langle\Omega_{AB}| \in \mathrm{Proj}\,(H_A \otimes H_B)$ with

$$|\Omega_{AB}\rangle = \frac{1}{\sqrt{2}} \sum_{i=1}^{2} |i_A\rangle \otimes |i_B\rangle.$$

Superdense coding is a protocol by which Alice can send 2 bits of classical information by transferring a qubit to Bob. Proceed as follows:

(1) Consider the Pauli matrices written as

$$\sigma_{00} = \mathbb{1}_2, \quad \sigma_{01} = \sigma_x, \quad \sigma_{10} = \sigma_y, \quad \sigma_{11} = \sigma_z.$$

Show that the vectors

$$|\psi_{ij}\rangle = (\mathbb{1}_2 \otimes \sigma_{ij})\,|\Omega_{AB}\rangle$$

define an orthonormal basis of $H_A \otimes H_B$. This orthonormal basis is called the "Bell basis".
(2) Find quantum channels $T_{ij} : B(H_A) \to B(H_A)$ such that

$$\mathrm{Tr}\left[|\psi_{kl}\rangle\langle\psi_{kl}|\,(T_{ij} \otimes \mathrm{id}_2)\,(\omega_{AB})\right] = \delta_{ki}\delta_{lj}.$$

for any $k, l, i, j \in \{0, 1\}$.
(3) Alice can use the quantum channels T_{ij} to "encode" 2 bits (i, j). If Alice transfers her qubit to Bob, then he can retrieve the message (i, j) by a measurement. How does this work in detail?

Excercise 4 (Quantum teleportation)**.** Consider again the two researchers Alice and Bob who each have a laboratory. Alice's laboratory contains two quantum systems labelled 'A' and '$\tilde{A}$' and Bob's laboratory contains a quantum system labelled 'B'. Each of these quantum systems is a qubit. Initially, 'A' and 'B' are in a maximally entangled pure state, i.e., their state is given by $\omega_{AB} = |\Omega_{AB}\rangle\langle\Omega_{AB}| \in \mathrm{Proj}\left(H_A \otimes H_B\right)$. Assume furthermore, that the system '$\tilde{A}$' is initially in some quantum state $\rho_{\tilde{A}} \in D(\mathcal{H}_{\tilde{A}})$ unknown to Alice and Bob. Quantum teleportation is a protocol that lets Alice send this unknown state to Bob by performing a measurement and sending the measurement outcome to Bob. Proceed as follows:

(1) Recall the Bell basis $\{|\psi_{ij}\rangle\}_{ij} \subset \mathbb{C}^2 \otimes \mathbb{C}^2$ from the previous exercise. Compute the probability of obtaining the outcome $(k,l) \in \{0,1\}^2$ when measuring the subsystems '$\tilde{A}A$' of the quantum state
$$\sigma_{ini} = \rho_{\tilde{A}} \otimes \omega_{AB},$$
using the PVM $\{|\psi_{ij}\rangle\langle\psi_{ij}|\}_{i,j}$ defined via the Bell basis.

(2) What is the post-measurement state σ_{post}^{ij} after obtaining outcome (i,j) in the measurement of (1)? **Hint:** It is very helpful to draw a diagram to see what is going on here!

(3) If Alice communicates the measurement outcome (i,j) of the Bell measurement to Bob, what quantum channel can he apply so that his system is in the final state $\rho_{\tilde{A}}$.

(4) How can we extend this protocol to quantum systems with dimension $d > 2$?

Excercise 5 (Bonus exercise)**.** Let $\rho_A \in D\left(\mathcal{H}_A\right)$ denote a quantum state with purification $|\psi_{AB}\rangle\langle\psi_{AB}| \in \mathrm{Proj}\left(\mathcal{H}_A \otimes \mathcal{H}_B\right)$, and consider a decomposition $\rho_A = \sum_{n=1}^N p_n \rho_n$ with quantum states $\rho_n \in D(\mathcal{H}_A)$ and a probability distribution $p \in \mathcal{P}\left(\{1,\ldots,N\}\right)$. We aim to construct an instrument $\{T_n\}_{n=1}^N$ with $T_n : B(\mathcal{H}_B) \to B(\mathcal{H}_B)$ such that
$$\mathrm{Tr}_B\left[\left(\mathrm{id}_A \otimes T_n\right)\left(|\psi_{AB}\rangle\langle\psi_{AB}|\right)\right] = p_n \rho_n,$$
for any $n \in \{1,\ldots,N\}$. Follow the steps below:

(1) Argue that without loss of generality we can assume $|\psi_{AB}\rangle = \mathrm{vec}\left(\sqrt{\rho_A}^T\right)$.

(2) Show that
$$\mathrm{Tr}_B\left[\left(\mathrm{id}_A \otimes L\right)\left(\mathrm{vec}\left(\sqrt{\rho}^T\right)\mathrm{vec}\left(\sqrt{\rho}^T\right)^\dagger\right)\right] = \sqrt{\rho} L^*\left(\mathbb{1}_{H_B}\right)^T \sqrt{\rho},$$
for any linear map $L : B(H_B) \to B(H_B)$ satisfying $L(X)^\dagger = L(X^\dagger)$ for any $X \in B(\mathcal{H}_B)$ and any $\rho \in D(\mathcal{H}_A)$.

(3) Find operators $K_n \in B(\mathcal{H}_B)$ for every $n \in \{1,\ldots,N\}$ such that
$$\sqrt{\rho_A}(K_n^\dagger K_n)^T \sqrt{\rho_A} = p_n \rho_n,$$
and such that $\sum_{n=1}^N K_n^\dagger K_n = \mathbb{1}_{\mathcal{H}_B}$.

Hint: Consider first the case where ρ_A is invertible and use the Moore-Penrose pseudoinverse for the general case. Note that we need to ensure that $\sum_{n=1}^N K_n^\dagger K_n = \mathbb{1}_{\mathcal{H}_B}$.

(4) Construct an instrument $\{T_n\}_{n=1}^N$ with $T_n : B(\mathcal{H}_B) \to B(\mathcal{H}_B)$ such that
$$\mathrm{Tr}_B\left[\left(\mathrm{id}_A \otimes T_n\right)\left(|\psi_{AB}\rangle\langle\psi_{AB}|\right)\right] = p_n \rho_n,$$
for any $n \in \{1,\ldots,N\}$.