EXERCISES 5

Excercise 1 (Matrix norms)**.** For a complex Euclidean space $\mathcal{H}$ consider an operator $A \in B(\mathcal{H})$ with singular values $s_1, \ldots, s_R \in \mathbb{R}^+$ ordered such that

$$s_1 \geqslant s_2 \geqslant \cdots \geqslant s_R.$$

For any $1 \leqslant p < \infty$ we define the matrix p-norm as

$$\|A\|_p = \mathrm{Tr}\left[|A|^p\right]^{1/p},$$

where $|A| = \sqrt{A^\dagger A}$.

(1) Show that

$$\|A\|_p = \left(\sum_{i=1}^{R} s_i^p\right)^{1/p},$$

for any $1 \leqslant p < \infty$.

(2) Show that we have

$$\|A\|_{HS} = \|A\|_2,$$

and that

$$\|A\|_\infty := \lim_{p\to\infty} \|A\|_p = s_1 = \|A\|.$$

(3) Show that

$$\|A\|_\infty \leqslant \|A\|_p \leqslant \|A\|_1,$$

for any $1 \leqslant p \leqslant \infty$.

(4) Show that

$$\|A\|_p = \sup_{\|B\|_q = 1} |\langle B, A\rangle_{HS}|,$$

whenever $1/p + 1/q = 1$. This means that $\|\cdot\|_p$ and $\|\cdot\|_q$ are dual norms for these values of p and q.

Excercise 2 (The fidelity)**.** Let $\mathcal{H}$ denote a complex Euclidean space. For quantum states $\rho, \sigma \in D(\mathcal{H})$, we define the *fidelity* by

$$F(\rho, \sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1,$$

where $\sqrt{\cdot}$ denotes the unique positive square root and $\|\cdot\|_1$ is the trace norm (i.e., the matrix 1-norm). The fidelity measures the distance between the two quantum states, and in the following we will prove some of its properties.

(1) Show that

$$F(\rho, \sigma) = \mathrm{Tr}\left[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right],$$

for all quantum states $\rho, \sigma \in D(\mathcal{H})$.

(2) Show that $F(\rho, \sigma) = F(\sigma, \rho)$ for all quantum states $\rho, \sigma \in D(\mathcal{H})$.

(3) Show $F(\rho, \sigma) \geqslant 0$ with equality if and only if $\rho\sigma = 0$.

(4) Show that $F(\rho, \sigma) \leqslant 1$ with equality if and only if $\rho = \sigma$.

(5) Show that $F(V\rho V^\dagger, V\sigma V^\dagger) = F(\rho, \sigma)$ for any isometry $V : \mathcal{H} \to \mathcal{H}'$ into a complex Euclidean space $\mathcal{H}'$.

(6) Show that $F(|\psi\rangle\langle\psi|, \sigma) = \sqrt{\langle\psi|\sigma|\psi\rangle}$ for any pure quantum state $|\psi\rangle\langle\psi| \in \mathrm{Proj}(\mathcal{H})$ and any $\sigma \in D(\mathcal{H})$.

Excercise 3 (Entanglement manipulation)**.** Let $H_{A_1}, H_{A_2}, H_{B_1}, H_{B_2}$ denote complex Euclidean spaces. A quantum channel

$$T : B(H_{A_1} \otimes H_{B_1}) \to B(H_{A_2} \otimes H_{B_2})$$

is called a *separable*[1] *quantum channel* if it can be written as

$$T = \sum_{n=1}^{N} S_A^{(n)} \otimes R_B^{(n)},$$

for completely positive maps $S_A^{(n)} : B(H_{A_1}) \to B(H_{A_2})$ and $R_B^{(n)} : B(H_{B_1}) \to B(H_{B_2})$. Let us denote the set of separable channels by

$$\mathrm{SepC}\left((H_{A_1}, H_{B_1}) \to (H_{A_2}, H_{B_2})\right).$$

(1) Show that $T(\rho_{AB}) \in \mathrm{Sep}(H_{A_2}, H_{B_2})$ for any $\rho_{AB} \in \mathrm{Sep}(H_{A_1}, H_{B_1})$ and any $\mathrm{SepC}((H_{A_1}, H_{B_1}) \to (H_{A_2}, H_{B_2}))$. The separable states are preserved under separable operations!

(2) Using the teleportation protocol, construct a separable quantum channel

$$T \in \mathrm{SepC}\left((\mathbb{C}^2 \otimes \mathbb{C}^2, \mathbb{C}^2) \to (\mathbb{C}^2 \otimes \mathbb{C}^2, \mathbb{C}^2)\right),$$

such that

$$T(\rho \otimes \omega_{AB}) = \frac{1}{4}(\mathbb{1}_{\mathbb{C}^2} \otimes \mathbb{1}_{\mathbb{C}^2}) \otimes \rho \in D(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2),$$

for any quantum state $\rho \in D(\mathbb{C}^2)$ and where $\omega_{AB} \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$ denotes the (normalized) maximally entangled state. How does this result generalize to dimensions $d > 2$.

(3) Show that any quantum states $\rho \in D(\mathbb{C}^d \otimes \mathbb{C}^d)$ can be written as

$$\rho = T(\omega_d),$$

for some separable quantum channel $T \in \mathrm{SepC}\left((\mathbb{C}^d, \mathbb{C}^d) \to (\mathbb{C}^d, \mathbb{C}^d)\right)$, and where $\omega_d \in D(\mathbb{C}^d \otimes \mathbb{C}^d)$ denotes the normalized maximally entangled state. This is why ω_d is called *maximally* entangled.

Excercise 4 (Bound entanglement)**.** In the following, we denote by $\omega_d \in \mathrm{Proj}\left(\mathbb{C}^d\right)$ the normalized maximally entangled state. We have seen, that ω_d can be transformed into any quantum state $\rho \in D(\mathbb{C}^d \otimes \mathbb{C}^d)$ under separable quantum channels. Here, we want to study the problem of obtaining ω_2 from other quantum states using separable quantum channels.

(1) Show that

$$F(\omega_d, \sigma) \leqslant \sqrt{\frac{1}{d}},$$

for the normalized maximally entangled state $\omega_d \in \mathrm{Proj}(\mathbb{C}^d \otimes \mathbb{C}^d)$ and any quantum state $\sigma \in D(\mathbb{C}^d \otimes \mathbb{C}^d) \cap \mathrm{PPT}\left(\mathbb{C}^d, \mathbb{C}^d\right)$ with positive partial transpose.

(2) Let $\sigma \in D(\mathbb{C}^d \otimes \mathbb{C}^d) \cap \mathrm{PPT}\left(\mathbb{C}^d, \mathbb{C}^d\right)$ denote a quantum state with positive partial transpose. Argue that there is no $n \in \mathbb{N}$ such that

$$T_n\left(\sigma^{\otimes n}\right) = \omega_2,$$

for any separable quantum channel $T_n \in \mathrm{SepC}\left(((\mathbb{C}^d)^{\otimes n}, (\mathbb{C}^d)^{\otimes n}) \to (\mathbb{C}^2, \mathbb{C}^2)\right)$. Is it possible that

$$F(\omega_2, T_n\left(\sigma^{\otimes n}\right)) \to 1,$$

as $n \to \infty$?

(3) Conclude that there exist quantum states, which are entangled but from which no maximally entangled state ω_2 can be constructed using separable quantum channels. Such quantum states are called *bound entangled*.

[1] Do not confuse this with separable quantum states!