EXERCISES 6

Excercise 1 (Further properties of the fidelity)**.** In the following, let $\mathcal{H}$ denote a complex Euclidean space. For positive operators $X, Y \in B(\mathcal{H})^+$ we define the fidelity by

$$F(X,Y) = \|X^{\frac{1}{2}} Y^{\frac{1}{2}}\|_1.$$

Show the following statements:

(1) For any $a, b \in \mathbb{R}^+$ and all normalized vectors $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ we have

$$\|a|\psi\rangle\langle\psi| - b|\phi\rangle\langle\phi|\|_1 = \sqrt{(a+b)^2 - 4ab|\langle\psi|\phi\rangle|^2}.$$

What has this to do with the fidelity?

(2) For all positive operators $X, Y \in B(\mathcal{H})^+$ we have $F(X, YXY) = \langle X, Y\rangle_{HS}$.

(3) Consider a quantum state $\rho \in D(\mathcal{H})$ and a positive operator $X \in B(\mathcal{H})^+$ satisfying $X \leqslant \mathbb{1}_{\mathcal{H}}$ and $\langle X, \rho\rangle_{HS} > 0$. Then, we have

$$F\left(\rho, \frac{\sqrt{X}\rho\sqrt{X}}{\langle X, \rho\rangle_{HS}}\right) \geqslant \sqrt{\langle X, \rho\rangle_{HS}}.$$

This is also called the "gentle measurement lemma". Can you explain why?

Excercise 2 (The completely depolarizing channel)**.** Consider the completely depolarizing channel $\Delta : B(\mathbb{C}^d) \to B(\mathbb{C}^d)$ given by

$$\Delta(X) = \mathrm{Tr}\,[X]\,\frac{\mathbb{1}_{\mathbb{C}^d}}{d}.$$

(1) Show that there exist unitary operators $\{U_1, \ldots, U_{d^2}\} \subset \mathcal{U}\left(\mathbb{C}^d\right)$ such that

$$\Delta = \frac{1}{d^2}\sum_{i=1}^{d^2} \mathrm{Ad}_{U_i}\,.$$

(2) Consider a quantum channel $T : B(\mathbb{C}^{d_A}) \to B(\mathbb{C}^{d_B})$. Show that there exists a pure quantum state $|\psi_{EB}\rangle\langle\psi_{EB}| \in \mathrm{Proj}\left(\mathbb{C}^{d_E} \otimes \mathbb{C}^{d_B}\right)$ for some $d_E \in \mathbb{N}$ such that

$$\frac{\mathbb{1}_{\mathbb{C}^{d_A}}}{d_A} \otimes \frac{\mathbb{1}_{\mathbb{C}^{d_E}}}{d_E} \otimes T(X) = \frac{1}{d_E^2 d_A^2}\sum_{i=1}^{d_E^2}\sum_{j=1}^{d_A^2} \mathrm{Ad}_{\left(U_i^A \otimes V_j^E \otimes \mathbb{1}_{d_B}\right)} \circ \mathrm{Ad}_W\left(X \otimes |\psi_{EB}\rangle\langle\psi_{EB}|\right),$$

for some unitaries $\{U_1^A, \ldots, U_{d_A^2}^A\} \subset \mathcal{U}\left(\mathbb{C}^{d_A}\right)$ and $\{V_1^E, \ldots, V_{d_E^2}^E\} \subset \mathcal{U}\left(\mathbb{C}^{d_E}\right)$, and some unitary $W \in \mathcal{U}\left(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_E} \otimes \mathbb{C}^{d_B}\right)$.

Excercise 3 (Uhlmann's theorem)**.** Let $\mathcal{H}_A, \mathcal{H}_E$ denote complex Euclidean spaces and $\rho, \sigma \in D(\mathcal{H}_A)$ two quantum states. Consider a normalized vector $|\psi_{EA}\rangle \in \mathcal{H}_E \otimes \mathcal{H}_A$ satisfying $\mathrm{Tr}_E\,[|\psi_{EA}\rangle\langle\psi_{EA}|] = \rho$, and show that

$$F(\rho, \sigma) = \max\{|\langle\psi_{EA}|\phi_{EA}\rangle| \;:\; \mathrm{Tr}_E\,[|\phi_{EA}\rangle\langle\phi_{EA}|] = \sigma\}.$$

Hint: You may use the data-processing inequality

$$F(T(\rho), T(\sigma)) \geqslant F(\rho, \sigma),$$

for all quantum states $\rho, \sigma \in D(\mathcal{H})$ and all quantum channels $T : B(\mathcal{H}) \to B(\mathcal{H}')$. Also remember, that there exists an isometry $V : \mathcal{H}_A \to \mathcal{H}_E$ such that $(V \otimes \mathbb{1}_{\mathcal{H}_A})\,\mathrm{vec}\left(\sqrt{\rho}\right) = |\psi_{EA}\rangle$ by using the Schmidt decomposition.

Excercise 4 (The Schur complement and operator inequalities)**.** Let $\mathcal{H}$ denote a complex Euclidean space. For operators $X, Y \in B(\mathcal{H})^+$ and $Z \in B(\mathcal{H})$ consider the block operator

$$M = \begin{pmatrix} X & Z \\ Z^\dagger & Y \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{H}).$$

(1) Show that $M \geqslant 0$ if and only if $\ker(Y) \subseteq \ker(Z)$ and $X \geqslant ZY^{-1}Z^{\dagger}$ (using the Moore-Penrose pseudoinverse).

Hint: You might need the operator

$$M = \begin{pmatrix} \mathbb{1}_{\mathcal{H}} & -ZY^{-1} \\ 0 & \mathbb{1}_{\mathcal{H}} \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{H}),$$

somewhere in the proof.

(2) Consider the case where $X = H^2$, $Z = H$ and $Y = \mathbb{1}_{\mathcal{H}}$ for some selfadjoint operator $H \in B(\mathcal{H})_{sa}$, and show that $M \in \mathrm{Sep}\left(\mathbb{C}^2, \mathcal{H}\right)$. Now, use the previous exercise to show that the operator inequality

$$P(H^2) \geqslant P(H)^2$$

holds for any unital positive map $P : B(\mathcal{H}) \to B(\mathcal{H})$. This inequality is called *Kadison's inequality*.

(3) Finally, consider the case where $X = A$, $Z = \mathbb{1}_{\mathcal{H}}$ and $Y = A^{-1}$ for some invertible positive operator $A \in B(\mathcal{H})^{++}$, and show that $M \in \mathrm{Sep}\left(\mathbb{C}^2, \mathcal{H}\right)$. Now, show that the operator inequality

$$P(A^{-1}) \geqslant P(A)^{-1}$$

holds for any unital positive map $P : B(\mathcal{H}) \to B(\mathcal{H})$. This inequality is called *Choi's inequality*.