EXERCISES 7

Excercise 1 (Properties of the von Neumann entropy)**.** For a quantum state $\rho \in D(\mathcal{H})$ we define the *von Neumann entropy* as

$$H(\rho) = -\operatorname{Tr}\left[\rho \log(\rho)\right],$$

where the logarithm is in base 2.

(1) Show that $0 \leqslant H(\rho) \leqslant \log\left(\dim(\mathcal{H})\right)$. For which quantum states do we have $H(\rho) = 0$ and for which quantum states do we have $H(\rho) = \log\left(\dim(\mathcal{H})\right)$?

(2) Assume that $\rho = \operatorname{Tr}_B\left[|\psi_{AB}\rangle\langle\psi_{AB}|\right]$ and $\sigma = \operatorname{Tr}_A\left[|\psi_{AB}\rangle\langle\psi_{AB}|\right]$ for some pure state $|\psi_{AB}\rangle \in H_A \otimes H_B$. Show that $H(\rho) = H(\sigma)$.

(3) Consider a probability distribution $p \in \mathcal{P}\left(\{1, \ldots, N\}\right)$ and quantum states $\rho_1, \ldots, \rho_N \in D(\mathcal{H})$. Show that

$$H\left(\sum_{i=1}^{N} p_i |i\rangle\langle i| \otimes \rho_i\right) = H(p) + \sum_{i=1}^{N} p_i H(\rho_i).$$

Does a similar statement hold for $H(\sum_{i=1}^{N} p_i \rho_i)$, when $\rho_i \rho_j = 0$ for $i \neq j$?

Excercise 2 (Typical projectors)**.** Consider a quantum state $\rho \in D(\mathcal{H})$ with spectral decomposition $\rho = \sum_{x=1}^{d} p(x) |v_x\rangle\langle v_x|$ where $p \in \mathcal{P}\left(\{1, \ldots, d\}\right)$ is a probability distribution and $|v_x\rangle \in \mathcal{H}$. Recall the ϵ-typical sequences $\mathcal{T}_{n,\epsilon}(p)$ from the first lecture, and define a projector

$$\Pi_{n,\epsilon} = \sum_{(x_1,\ldots,x_n) \in \mathcal{T}_{n,\epsilon}(p)} |v_{x_1}\rangle\langle v_{x_1}| \otimes \cdots \otimes |v_{x_n}\rangle\langle v_{x_n}| \in \operatorname{Proj}\left(\mathcal{H}^{\otimes n}\right).$$

Show the following statements:

(1) For any $\epsilon > 0$ we have

$$\operatorname{Tr}\left[\Pi_{n,\epsilon} \rho^{\otimes n}\right] \to 1,$$

as $n \to \infty$.

(2) We have

$$\operatorname{Tr}\left[\Pi_{n,\epsilon}\right] < 2^{nH(\rho)+\epsilon}.$$

(3) For every $n \in \mathbb{N}$ let $\tilde{\Pi}_n \in \operatorname{Proj}\left(\mathcal{H}^{\otimes n}\right)$ denote a projector such that $\operatorname{rk}\left(\tilde{\Pi}_n\right) \leqslant 2^{nR}$ for some $R < H(\rho)$. Show that

$$\operatorname{Tr}\left[\tilde{\Pi}_n \rho^{\otimes n}\right] \to 0,$$

as $n \to \infty$.

Excercise 3 (Purifications and extensions)**.** For a quantum state $\rho \in D(\mathcal{H})$ consider a pure state $|\psi_{EA}\rangle \in \mathcal{H}_E \otimes \mathcal{H}_A$ such that $\operatorname{Tr}_E\left[|\psi_{EA}\rangle\langle\psi_{EA}|\right] = \rho$. Moreover, consider a quantum state $\sigma_{E'A} \in D(\mathcal{H}_{E'} \otimes \mathcal{H}_A)$ satisfying $\operatorname{Tr}_{E'}\left[\sigma_{E'A}\right] = \rho$. Show that there exists a quantum channel $S : B(\mathcal{H}_E) \to B(\mathcal{H}_{E'})$ such that

$$\left(S \otimes \mathrm{id}_A\right)\left(|\psi_{EA}\rangle\langle\psi_{EA}|\right) = \sigma_{E'A}.$$

Hint: Express the statement using the Choi-Jamiolkowski isomorphism and assume first that ρ is invertible.

Excercise 4 (Channel fidelity)**.** Let $\mathcal{H}$ denote a complex Euclidean space. For a quantum channel $T : B(\mathcal{H}) \to B(\mathcal{H})$ and a quantum state $\rho \in D(\mathcal{H})$ we define the *channel fidelity* of T with respect to ρ as

$$F(T, \rho) = F\left(\left(\mathrm{id}_E \otimes T\right)\left(|\psi_{EA}\rangle\langle\psi_{EA}|\right), |\psi_{EA}\rangle\langle\psi_{EA}|\right),$$

where $|\psi_{EA}\rangle = \operatorname{vec}\left(\sqrt{\rho}\right)$. Show the following properties of this quantity:

(1) Assume that $T = \sum_{n=1}^{N} \mathrm{Ad}_{K_n}$ for Kraus operators $K_n \in B(\mathcal{H})$. Show that

$$F(T, \rho) = \sqrt{\sum_{n=1}^{N} |\langle \rho, K_n \rangle_{HS}|^2},$$

for any quantum state $\rho \in D(\mathcal{H})$.

(2) Consider an invertible quantum state $\rho \in D(\mathcal{H})$. Show that $F(T, \rho) = 1$ implies that $T = \mathrm{id}$. Is this also true when ρ is not invertible?

(3) Consider a quantum state $\sigma_{E'A} \in D(\mathcal{H}_{E'} \otimes \mathcal{H}_A)$ satisfying $\mathrm{Tr}_{E'} [\sigma_{E'A}] = \rho$. Show that

$$F\left(\left(\mathrm{id}_{E'} \otimes T \right) \left(\sigma_{E'A} \right), \sigma_{E'A} \right) \geqslant F(T, \rho).$$

What does this imply for the quantum channel T when $F(T, \rho)$ is close to 1?