EXERCISES 9

Excercise 1. For a quantum state $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ we define the *mutual information* as

$$I(A:B)_{\rho_{AB}} = H(\rho_A) + H(\rho_B) - H(\rho_{AB}).$$

(1) Verify that
$$I(A:B)_{\rho_{AB}} = D(\rho_{AB} \| \rho_A \otimes \rho_B),$$
for any quantum state $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$.

(2) Show that
$$I(A:B)_{\rho_{AB}} \leqslant I(A:(B,C))_{\rho_{ABC}},$$
for any quantum state $\rho_{ABC} \in D(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$.

Excercise 2 (Partial convexity of Holevo's quantity)**.** Consider two ensembles $\{p(x), \rho_x\}_{x\in\Sigma}$ and $\{q(x), \sigma_x\}_{x\in\Sigma}$ such that

$$\sum_{x\in\Sigma} p(x)\rho_x = \sum_{x\in\Sigma} q(x)\sigma_x,$$

and $p(x) \neq 0$ and $q(x) \neq 0$ for any $x \in \Sigma$. Show that

$$\chi\left(\{(1-\lambda)p(x) + \lambda q(x), \frac{(1-\lambda)p(x)\rho_x + \lambda q(x)\sigma_x}{(1-\lambda)p(x) + \lambda q(x)}\}\right) \leqslant (1-\lambda)\chi\left(\{p(x), \rho_x\}\right) + \lambda\chi\left(\{q(x), \sigma_x\}\right).$$

Does this inequality also hold for general ensembles of quantum states?

Excercise 3 (An operator inequality)**.** Here, we will prove that the inequality

$$\mathbb{1}_{\mathcal{H}} - (X+Y)^{-\frac{1}{2}} X (X+Y)^{-\frac{1}{2}} \leqslant 2(\mathbb{1}_{\mathcal{H}} - X) + 4Y, \tag{1}$$

holds for any $X, Y \in B(\mathcal{H})^+$ such that $X \leqslant \mathbb{1}_{\mathcal{H}}$.

(1) For $X, Y \in B(\mathcal{H})^+$ show that $X^{\frac{1}{2}} \leqslant (X+Y)^{\frac{1}{2}}$.

(2) Show that
$$ZY(\mathbb{1}_{\mathcal{H}} - Z)^\dagger + (\mathbb{1}_{\mathcal{H}} - Z)YZ^\dagger \leqslant ZYZ^\dagger + (\mathbb{1}_{\mathcal{H}} - Z)Y(\mathbb{1}_{\mathcal{H}} - Z)^\dagger,$$
holds for any $Y \in B(\mathcal{H})^+$ and any $Z \in B(\mathcal{H})$.

(3) Choose $Z = (X+Y)^{\frac{1}{2}}$ and use the previous exercise to show that
$$Y \leqslant 2(X+Y)^{\frac{1}{2}} Y (X+Y)^{\frac{1}{2}} + 2\left(\mathbb{1}_{\mathcal{H}} - (X+Y)^{\frac{1}{2}}\right)(X+Y)\left(\mathbb{1}_{\mathcal{H}} - (X+Y)^{\frac{1}{2}}\right).$$
Hint: Start by expanding $(\mathbb{1}_{\mathcal{H}} - Z + Z)Y(\mathbb{1}_{\mathcal{H}} - Z + Z)$.

(4) Finally, show that
$$Y \leqslant (X+Y)^{\frac{1}{2}}(2\mathbb{1}_{\mathcal{H}} - 2X + 4Y)(X+Y)^{\frac{1}{2}},$$
for any $X, Y \in B(\mathcal{H})^+$, and conclude that (1) holds.
Hint: You might need that $X \leqslant X^{\frac{1}{2}}$ for any $X \in B(\mathcal{H})^+$ satisfying $X \leqslant \mathbb{1}_{\mathcal{H}}$.

Excercise 4 (Pinsker's inequality)**.** Here, we want to prove the following statement: For any quantum states $\rho, \sigma \in D(\mathcal{H})$, we have

$$D(\rho\|\sigma) \geqslant \frac{1}{2\ln(2)}\|\rho - \sigma\|_1^2.$$

You can find your own proof of this statement, or follow the steps below:

(1) Show that for any $a, b \in [0,1]$ we have

$$a \ln\left(\frac{a}{b}\right) + (1-a)\ln\left(\frac{1-a}{1-b}\right) \geqslant 2(a-b)^2.$$

Hint: To show the statement for $a, b \in (0,1)$ prove the statement

$$a \ln\left(\frac{a}{b}\right) + (1-a)\ln\left(\frac{1-a}{1-b}\right) = f(b) - f(a) + (a-b)f'(b),$$

for $f : [0,1] \to \mathbb{R}$ given by $f(x) = -x\ln(x) - (1-x)\ln(1-x)$, and then use the Taylor expansion of f.

(2) Prove the classical version of Pinsker's inequality: For any $p, q \in \mathcal{P}(\{1,\ldots,d\})$ we have

$$D(p\|q) \geqslant \frac{1}{2\ln(2)}\|p-q\|_1^2,$$

where $\|\cdot\|_1$ denotes the statistical distance.
Hint: Consider the following sets:

$$\begin{aligned} I_0 &= \{i \in \{1,\ldots,d\} \ : \ p(i) > q(i)\} \\ I_1 &= \{i \in \{1,\ldots,d\} \ : \ p(i) < q(i)\} \\ I_2 &= \{i \in \{1,\ldots,d\} \ : \ p(i) = q(i)\}, \end{aligned}$$

and define a classical channel $A : \{1,\ldots,d\} \to \mathcal{P}(\{0,1\})$ such that

$$A(i) = \begin{cases} \delta_0, & \text{if } i \in I_0 \\ \delta_1, & \text{if } i \in I_1 \\ \frac{1}{2}(\delta_0 + \delta_1), & \text{if } i \in I_2, \end{cases}$$

where $\delta_j \in \mathcal{P}(\{0,1\})$ denotes the deterministic probability distribution with outcome j. Then, use the data-processing inequality.

(3) Finally, consider a POVM discriminating the states ρ and σ and achieving the Holevo-Helstrom bound. Use the data-processing inequality for the corresponding quantum-to-classical channel to reduce the quantum Pinsker's inequality to the classical case.