① Schatten p-norms :

$$\|A\|_p = \mathrm{Tr}\left[|A|^p\right]^{\frac{1}{p}} \quad , \quad |A| = \sqrt{A^\dagger A}$$

(1) Singular value decomposition $\Longrightarrow$ $A = USV$, $U, V \in \mathcal{U}(H)$

$$S = \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_{\dim(H)} \end{pmatrix}$$

$$|A| = \sqrt{A^\dagger A} = \sqrt{V^\dagger S U^\dagger U S V}$$
$$= \sqrt{V^\dagger S^2 V}$$
$$= V^\dagger S V$$

$$\Longrightarrow \|A\|_p = \mathrm{Tr}\left[|A|^p\right]^{\frac{1}{p}} = \mathrm{Tr}\left[V^\dagger S^p V\right]^{\frac{1}{p}} = \left(\sum_i s_i(A)^p\right)^{\frac{1}{p}}.$$

$\|A\|_p$ coincides with the p-norm of the vector of singular values!

(2) $\|A\|_{HS} = \sqrt{\langle A, A\rangle_{HS}}$

$= \sqrt{\mathrm{Tr}[A^\dagger A]}$

$= \sqrt{\mathrm{Tr}[|A|^2]} = \|A\|_2 .$

Recall from usual p-norms

$$s \in \mathbb{R}^d \Longrightarrow \lim_{p\to\infty} \|s\|_p = \|s\|_\infty = \max_{1\le i\le d} |s_i|$$

Proof: If $s_1 \ge s_k \ \forall k \in \{1,\dots,d\}$, then we have

$$\|s\|_p = \Big(\sum_i s_i^p\Big)^{\frac{1}{p}} = s_1 \Big(\sum_i \Big(\frac{s_i}{s_1}\Big)^p\Big)^{\frac{1}{p}} = s_1 \exp\Big(\underbrace{\frac{1}{p}\log\Big(\sum_i \Big(\frac{s_i}{s_1}\Big)^p\Big)}_{\longrightarrow 0 \text{ as } p\to\infty}\Big)$$

$$\longrightarrow s_1 \quad (p\to\infty).$$

□

We have:

$$\|A\|_p = \|s\|_p \longrightarrow \|s\|_\infty = \max_{1\le i\le d} |s_i| = \|A\|_\infty .$$

□

(3) Recall ordering of usual p-norms:

$$s \in \mathbb{R}^d \implies \|s\|_\infty \leq \|s\|_p \leq \|s\|_1 \quad \forall p \in [1, \infty].$$

Proof: Note that $\|s\|_p \geq \|s\|_\infty \ \forall p \in [1,\infty]$, and therefore

$$\frac{s_i}{\|s\|_q} \leq 1 \quad \forall i \in \{1, \dots, d\} \quad \forall q \in [1, \infty).$$

If $q \leq p$, then we have

$$\frac{\|s\|_p^p}{\|s\|_q^p} = \sum_i \left(\frac{s_i}{\|s\|_q}\right)^p \underset{q \leq p}{\leq} \sum_i \left(\frac{s_i}{\|s\|_q}\right)^q = 1$$

□

We conclude that

$$\|A\|_\infty \leq \|A\|_p \leq \|A\|_1 \quad \forall p \in [1, \infty].$$

(4) We need the following:

Thm (Hölder's inequality):

For $A, B \in \mathcal{B}(H)$ we have

$$|\langle A, B \rangle_{HS}| \leq \|A\|_p \, \|B\|_q \quad \text{when} \quad \frac{1}{p} + \frac{1}{q} = 1$$

Proof: By singular value decomposition $\exists\, U, V \in \mathcal{U}(H)$ s.t.

$$|\langle A, B \rangle_{HS}| = |\mathrm{Tr}[A^+ B]|$$

$$= |\mathrm{Tr}[USVR]| \quad \text{where} \quad S = \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_d \end{pmatrix} \text{ singular values of } A$$

$$R = \begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_d \end{pmatrix} \text{ singular values of } B$$

we have:

$$|\mathrm{Tr}[USVR]| = \left|\mathrm{Tr}\left[\left(R^{\frac{1}{2}} U S^{\frac{1}{2}}\right)\left(S^{\frac{1}{2}} V R^{\frac{1}{2}}\right)\right]\right|$$

$$= |\langle S^{\frac{1}{2}} U^+ R^{\frac{1}{2}}, S^{\frac{1}{2}} V R^{\frac{1}{2}} \rangle_{HS}|$$

$$\overset{\text{Cauchy-Schwarz}}{\leq} \|S^{\frac{1}{2}} U^+ R^{\frac{1}{2}}\|_2 \, \|S^{\frac{1}{2}} V R^{\frac{1}{2}}\|_2$$

$$= \mathrm{Tr}[U^+ R U S]^{\frac{1}{2}} \mathrm{Tr}[V R V^+ S]^{\frac{1}{2}}.$$

$$|\langle A,B\rangle_{HS}| \leq \mathrm{Tr}\left[U^\dagger R U S\right]^{\frac12}\mathrm{Tr}\left[V R V^\dagger S\right]^{\frac12}$$

$S=\begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_d \end{pmatrix}$ singular values of A

$R=\begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_d \end{pmatrix}$ singular values of B

$$\mathrm{Tr}\left[U^\dagger R U S\right] = \sum_{ij} r_i s_j \,\mathrm{Tr}\left[|u_i\rangle\langle u_i| \, |j\rangle\langle j|\right]$$

$$= \sum_{ij} r_i s_j \, |\langle u_i|j\rangle|^2$$

$\frac{1}{p}+\frac{1}{q}=1$

$$= \sum_{ij} r_i \left(|\langle u_i|j\rangle|^2\right)^{\frac1q} s_j \left(|\langle u_i|j\rangle|\right)^{\frac1p}$$

Hölder

$$\leq \left(\sum_{ij} r_i^q |\langle u_i|j\rangle|^2\right)^{\frac1q}\left(\sum_{ij} s_j^p |\langle u_i|j\rangle|\right)^{\frac1p}$$

$\sum_i |\langle u_i|j\rangle|^2 = \sum_j |\langle u_i|j\rangle|^2 = 1$

$$= \|r\|_q \|s\|_p = \|A\|_p \|B\|_q .$$

Same argument:

$$\mathrm{Tr}\left[V R V^\dagger S\right] \leq \|A\|_p \|B\|_q . \quad \Longrightarrow \quad |\langle A,B\rangle_{HS}|$$

$$\leq \|A\|_p^{\frac12} \|B\|_q^{\frac12} \|A\|_p^{\frac12} \|B\|_q^{\frac12}$$

$$= \|A\|_p \|B\|_q . \qquad \square$$

Now, we can show:

$$\|A\|_p = \sup_{\|B\|_q = 1} |\langle B, A\rangle_{HS}| .$$

Proof: By Hölder's inequality it is clear that

$$\sup_{\|B\|_q = 1} |\langle B, A\rangle_{HS}| \leq \|A\|_p .$$

To show that supremum is attained consider $r \in \mathbb{R}^d$ s.th.

$$\begin{cases} \|r\|_q = 1 \\ \& \ \langle r, s\rangle = \|s\|_p \end{cases} \qquad (*)$$

SVD: $A = USV$ $\rightsquigarrow$ Choose $B = URV$ for $R = \begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_d \end{pmatrix}$

Then:

$$\begin{aligned} |\langle B, A\rangle_{HS}| = \mathrm{Tr}[B^+ A] = \mathrm{Tr}[B^+ USV] \\ = \mathrm{Tr}[(U^+ B V^+)^+ S] \\ = \mathrm{Tr}[RS] = \langle r, s\rangle = \|s\|_p = \|A\|_p . \end{aligned}$$

To show (*):

Consider $t_i = s_i^{p-1}$ and check

$$\|t\|_q^q = \sum_i s_i^{qp-q} = \sum_i s_i^p = \|s\|_p^p. \Longrightarrow \|t\|_q = \|s\|_p^{p/q} = \|s\|_p^{p-1}$$

$\uparrow$ $\frac{1}{p}+\frac{1}{q}=1$

$\Leftrightarrow p+q=pq$

since

$$\frac{p}{q} = p\left(1-\frac{1}{p}\right) = p-1.$$

Set $r_i = \frac{t_i}{\|t\|_q} \Longrightarrow \|r\|_q = 1$

&

$$\sum_i r_i \cdot s_i = \sum_i \frac{s_i^p}{\|t\|_q} = \frac{\|s\|_p^p}{\|s\|_p^{p-1}} = \|s\|_p.$$

□

② Fidelity : $F(\rho,\sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1$

(1) $\|A\|_1 = \mathrm{Tr}[|A|] = \mathrm{Tr}\left[\sqrt{A^\dagger A}\right] = \mathrm{Tr}\left[\sqrt{AA^\dagger}\right]$.

$$\Rightarrow F(\rho,\sigma) = \mathrm{Tr}\left[\sqrt{\sqrt{\rho}\sqrt{\sigma}\sqrt{\sigma}\sqrt{\rho}}\right]$$
$$= \mathrm{Tr}\left[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right].$$

(2) $F(\rho,\sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1 = \|(\sqrt{\rho}\sqrt{\sigma})^\dagger\|_1 = F(\sigma,\rho)$.

(3) $F(\rho,\sigma) \geq 0$ Clear !

$F(\rho,\sigma) = 0 \iff \sqrt{\rho}\sqrt{\sigma} = 0 \iff \rho\sigma = 0$.

(4) SVD $\Rightarrow \sqrt{\rho}\sqrt{\sigma} = USV$, $U, V \in \mathcal{U}(H)$, $S = \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_d \end{pmatrix}$

$$\|\sqrt{\rho}\sqrt{\sigma}\|_1 = \mathrm{Tr}\left[V^\dagger U^\dagger \sqrt{\rho}\sqrt{\sigma}\right]$$
$$= \langle \sqrt{\rho}W, \sqrt{\sigma}\rangle_{HS} \leq \|\sqrt{\rho}W\|_2 \|\sqrt{\sigma}\|_2 = 1.$$

$W = UV$ ↗

↑ Cauchy-Schwarz

Equality in Cauchy-Schwarz

$\Longleftrightarrow \sqrt{\rho}W = \lambda\sqrt{\sigma}$

$\Longleftrightarrow \rho = \lambda^2\sigma \overset{\mathrm{Tr}(\rho)=\lambda^2=1}{\Longleftrightarrow} \rho = \sigma$ □

(5) $V: H \to H'$ isometry

$\Longrightarrow V^\dagger V = \mathbb{1}_H$.

$$F(V\rho V^\dagger, V\sigma V^\dagger) = \| \sqrt{V\rho V^\dagger}\sqrt{V\sigma V^\dagger} \|_1$$

$$= \| V\sqrt{\rho}V^\dagger V\sqrt{\sigma}V^\dagger \|_1$$

$$= \| V\sqrt{\rho}\sqrt{\sigma}V^\dagger \|_1 = \| \sqrt{\rho}\sqrt{\sigma} \|_1$$

$\sqrt{V\rho V^\dagger} = V\sqrt{\rho}V^\dagger$

Singular values invariant under isometries.

(6) $F(|\psi\rangle\langle\psi|, \sigma) = \| |\psi\rangle\langle\psi| \sqrt{\sigma} \|_1$

$$= \mathrm{Tr}\left[\sqrt{|\psi\rangle\langle\psi|\sqrt{\sigma}\sqrt{\sigma}|\psi\rangle\langle\psi|}\right] = \sqrt{\langle\psi|\sigma|\psi\rangle}\,\mathrm{Tr}\left[|\psi\rangle\langle\psi|\right] = \sqrt{\langle\psi|\sigma|\psi\rangle}$$

($\sqrt{|\psi\rangle\langle\psi|\sqrt{\sigma}\sqrt{\sigma}|\psi\rangle\langle\psi|} = \langle\psi|\sigma|\psi\rangle$)

□

Motivation: The resource theory of entanglement

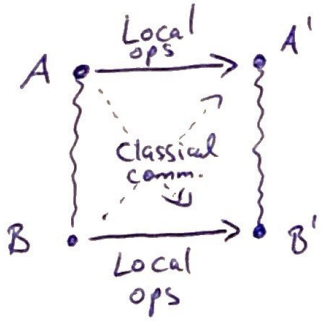
Setting: Bipartite quantum system $H_A \otimes H_B$

↪ Idea: "A" and "B" are far apart!

⇒ They can act only locally on their systems.

↪ Apply local quantum channels
Measure local subsystems.

& exchange classical information (e.g. outcomes of measurements) via a telephone line.



↓

Local Operations and Classical Communication

LOCC.

Which quantum channels

$$T: \mathcal{B}(H_A \otimes H_B) \longrightarrow \mathcal{B}(H_{A'} \otimes H_{B'})$$

are LOCC?

Answer is complicated!

But: $\mathrm{LOCC}((H_A, H_B) \to (H_{A'}, H_{B'})) \subsetneq \mathrm{SepC}((H_A, H_B) \to (H_{A'}, H_{B'}))$

Separable quantum channels:

$$T = \sum_{n=1}^{N} S_A^{(n)} \otimes R_B^{(n)}$$

for completely positive maps

$$S_A^{(n)} : \mathcal{B}(H_A) \to \mathcal{B}(H_{A'})$$

$$R_B^{(n)} : \mathcal{B}(H_B) \to \mathcal{B}(H_{B'}) .$$

To understand LOCC, we can understand SepC !

③ (1) Separable channels preserve separable states !

$$\rho_{AB} = \sum_i p_i \, \sigma_i^A \otimes \tau_i^B \quad \in D(H_{A_1} \otimes H_{B_1}) \cap Sep(H_{A_1}, H_{B_1})$$

$$T \in SepC(H_{A_1}, H_{B_1}) \longrightarrow (H_{A_2}, H_{B_2}))$$

$$\Longrightarrow \quad T(\rho_{AB}) = \sum_{in} p_i \underbrace{S_A^{(n)}(\sigma_i^A)}_{\geq 0} \otimes \underbrace{R_B^{(n)}(\tau_i^B)}_{\geq 0} \in Sep(H_{A_2}, H_{B_2})$$

$\rightsquigarrow$ separable quantum state.

(2) Let $|\psi_i\rangle$ $i=1,2,3,4$ denote the Bell basis.

Define $T \in \mathrm{Sep}\,\mathbb{C}((\mathbb{C}^2 \otimes \mathbb{C}^2, \mathbb{C}^2) \longrightarrow (\mathbb{C}^2 \otimes \mathbb{C}^2, \mathbb{C}^2))$

by $T = \sum_{i=1}^{4} \mathrm{Ad}_{|\psi_i\rangle\langle\psi_i|} \otimes \mathrm{Ad}_{\sigma_i}$ $\rightsquigarrow$ clearly separable!

$$\Rightarrow \quad T(\rho_{A'} \otimes \omega_{AB})$$

$$= \sum_i \left(|\psi_i^{AA'}\rangle\langle\psi_i^{AA'}| \otimes \sigma_i\right)(\rho_{A'} \otimes \omega_{AB})\left(|\psi_i^{AA'}\rangle\langle\psi_i^{AA'}| \otimes \sigma_i\right)$$

$$= \frac{1}{4}\sum_i |\psi_i^{AA'}\rangle\langle\psi_i^{AA'}| \otimes \mathrm{Ad}_{\sigma_i}(\sigma_i \rho_B \sigma_i)$$

$$= \frac{1}{4}\sum_i |\psi_i^{AA'}\rangle\langle\psi_i^{AA'}| \otimes \rho_B = \frac{\mathbb{1}}{2} \otimes \frac{\mathbb{1}}{2} \otimes \rho_B .$$

$\rightsquigarrow$ This channel is even LOCC!

$\rightsquigarrow$ Generalizes to $d>2$ by using OTS of unitary operators.

(3) Consider:

(i) $S_S \in Sep\mathbb{C}\left((\mathbb{C}^d, \mathbb{C}^d) \longrightarrow (\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d, \mathbb{C}^d)\right)$

$$S_S(X_{AB}) = S_{RA'} \otimes X_{AB}$$

(ii) $T \in Sep\mathbb{C}\left((\mathbb{C}^d \otimes \mathbb{C}^d, \mathbb{C}^d) \longrightarrow (\mathbb{C}^d \otimes \mathbb{C}^d, \mathbb{C}^d)\right)$

Teleportation channel from before!

$\Longrightarrow (id_R \otimes T) \in Sep\mathbb{C}\left((\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d, \mathbb{C}^d) \longrightarrow (\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d, \mathbb{C}^d)\right)$

(iii) Partial trace ($\rightsquigarrow$ discarding local systems)

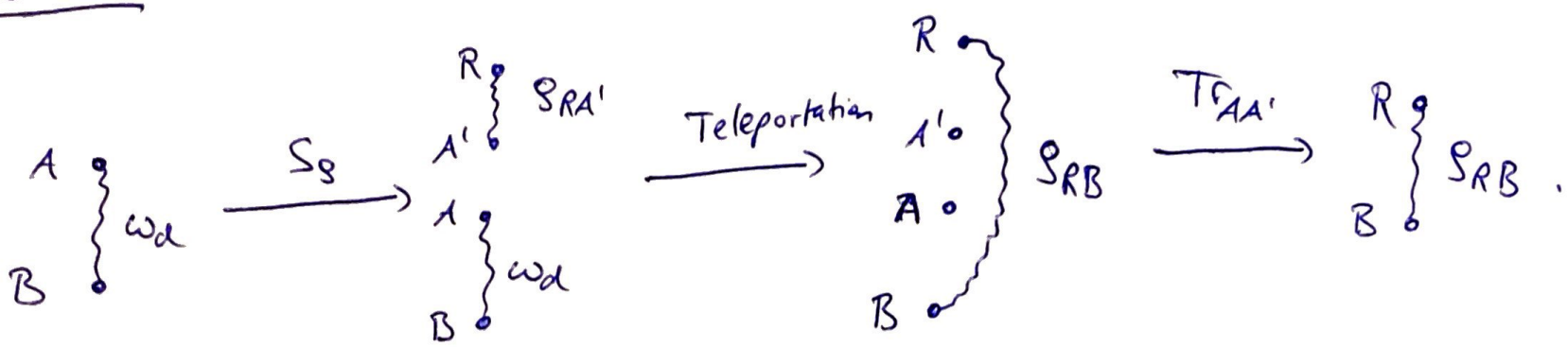
$$Tr_{AA'} = id_R \otimes Tr_{AA'} \otimes id_B .$$

Composition: $Tr_{AA'} \circ (id_R \otimes T) \circ S_S \in Sep\mathbb{C}\left((\mathbb{C}^d, \mathbb{C}^d) \longrightarrow (\mathbb{C}^d, \mathbb{C}^d)\right)$

Satisfies:

$$Tr_{AA'} \circ (id_R \otimes T) \circ S_\rho(\omega_d) = \rho_{RB} .$$

Picture:



From ω_d we can get any entangled state!

$\Longrightarrow$ ω_d is maximally entangled!

(4) Bound entanglement.

(1) $\sigma \in D(\mathbb{C}^d \otimes \mathbb{C}^d) \cap PPT(\mathbb{C}^d, \mathbb{C}^d)$

$$\Longrightarrow F(\omega_d, \sigma) = \sqrt{\langle \Omega_d | \sigma | \Omega_d \rangle}$$

$$= \sqrt{\mathrm{Tr}\left[|\Omega_d\rangle\langle\Omega_d| \, \sigma \right]}$$

$$= \sqrt{\frac{1}{d} \mathrm{Tr}\left[\mathbb{F} \sigma^\Gamma \right]} \leq \frac{1}{\sqrt{d}}$$

$\mathbb{F} = C_\varphi$ is a unitary!

$$\mathrm{Tr}\left[\mathbb{F}\sigma^\Gamma\right] = \langle \mathbb{F}, \sigma^\Gamma \rangle_{HS}$$

$$\overset{\text{Hölder}}{\leq} \|\mathbb{F}\|_\infty \|\underbrace{\sigma^\Gamma}_{\text{quantum state}}\|_1 = 1 .$$

$\rightsquigarrow$ PPT states are far from maximally entangled!

(2) We need the following lemma:

Lemma: If $R : \mathcal{B}(\mathbb{C}^d) \to \mathcal{B}(\mathbb{C}^{d'})$ is completely positive, then there exists completely positive map $R' : \mathcal{B}(\mathbb{C}^d) \to \mathcal{B}(\mathbb{C}^{d'})$ s.t.

$$\mathcal{V} \circ R = R' \circ \mathcal{V}$$

Proof: Note that

$$(AXA^*)^T = \bar{A} X^T A^T$$

$$\Rightarrow \quad \mathcal{V} \circ Ad_A = Ad_{\bar{A}} \circ \mathcal{V}$$

Now, Lemma follows by Kraus decomposition.

□

Consequence:

Separable quantum channels preserve PPT quantum states!

$$\sigma \in \mathcal{D}(\mathbb{C}^q \otimes \mathbb{C}^q) \cap \mathrm{PPT}(\mathbb{C}^q, \mathbb{C}^q)$$

$$T = \sum_n S_A^{(n)} \otimes R_B^{(n)} \in \mathrm{Sep}\left((\mathbb{C}^q, \mathbb{C}^q) \longrightarrow (\mathbb{C}^{q'}, \mathbb{C}^{q'})\right)$$

Then:

$$T(\sigma)^\Gamma = \sum_n \left(S_A^{(n)} \otimes \vartheta \circ R_B^{(n)}\right)(\sigma)$$

$$= \sum_n \underbrace{\left(S_A^{(n)} \otimes \tilde{R}_B^{(n)}\right)(\sigma^\Gamma)}_{\geq 0} \geqslant 0 \,.$$

$\sigma^\Gamma \geqslant 0$

$$\Longrightarrow T(\sigma) \in \mathcal{D}(\mathbb{C}^q \otimes \mathbb{C}^q) \cap \mathrm{PPT}(\mathbb{C}^q, \mathbb{C}^q) \,.$$

Another lemma:

Lemma: If $\sigma \in D(\mathbb{C}^q \otimes \mathbb{C}^q) \cap PPT(\mathbb{C}^q, \mathbb{C}^q)$, then

$$\sigma^{\otimes n} \in D((\mathbb{C}^q)^{\otimes n} \otimes (\mathbb{C}^q)^{\otimes n}) \cap PPT((\mathbb{C}^q)^{\otimes n}, (\mathbb{C}^q)^{\otimes n})$$

Proof: $2_d^{\otimes n}$ is the transpose on $B((\mathbb{C}^q)^{\otimes n})$

$\rightsquigarrow$ Check on computational basis.

$$(\sigma^{\otimes n})^\Gamma = (\sigma^\Gamma)^{\otimes n}.$$

□

Now for the final argument:

$$\sigma \in D(\mathbb{C}^q \otimes \mathbb{C}^q) \cap PPT(\mathbb{C}^q, \mathbb{C}^q)$$

$$\Longrightarrow \sigma^{\otimes n} \text{ PPT state} \longrightarrow T_n(\sigma^{\otimes n}) \text{ PPT state} \Longrightarrow F(T_n(\sigma^{\otimes n}), \omega_2) \leq \frac{1}{\sqrt{2}}$$

$\forall T_n$ separable quantum channel

$$F(T_n(\sigma^{\otimes n}), \omega_2) \leq \frac{1}{\sqrt{2}} \ll 1.$$

$\Longrightarrow$ It is impossible that

$$T_n(\sigma^{\otimes n}) \longrightarrow \omega_2 \quad \text{as} \quad n \to \infty.$$

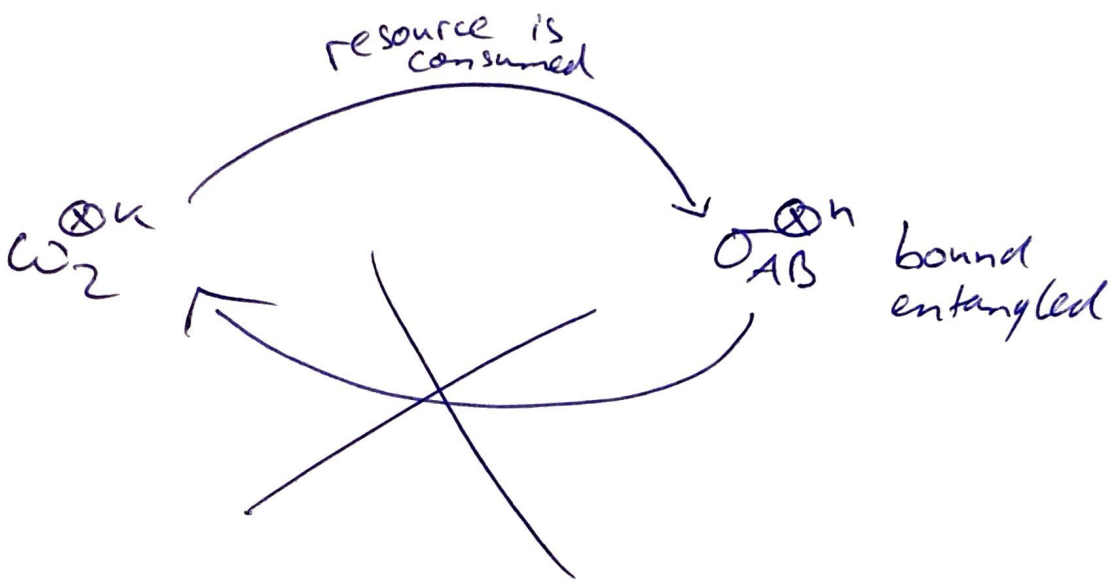
~~σ is bound ent~~

(4) If σ_{AB} is entangled PPT state, then it is impossible to obtain ω_2 as a limit of states $T_n(\sigma_{AB}^{\otimes n})$ for $n \to \infty$.

$\rightsquigarrow$ σ is <u>bound entangled</u>.

Fundamental asymmetry:

If we consider ω_2 as a resource, then we have:



We cannot get the resource back!

under separable quantum channels.