**Quantum information theory (MAT4430)** **Spring 2021**

# Note on quantum teleportation

*Lecturer: Alexander Müller-Hermes*

Here, we will use quantum entanglement and classical information to send qubits perfectly. The basic protocol to do this is called the *teleportation protocol*. It works like this: Alice and Bob have their respective laboratories at some distance from each other, and in each of the laboratories they have a qubit (with respective state spaces $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$) that they can access and manipulate. Again, the qubits are initially in the maximally entangled state $|\Omega_{AB}\rangle\langle\Omega_{AB}| \in \mathrm{Proj}\left(\mathcal{H}_A \otimes \mathcal{H}_B\right)$ from the previous section. Alice has another qubit with corresponding state space $\mathcal{H}_{A'} = \mathbb{C}^2$ in some unknown quantum state $\rho_{A'} \in D(\mathcal{H}_{A'})$. Using the quantum teleportation protocol, Alice can send the quantum state $\rho_{A'}$ to Bob by applying a quantum channel to the quantum systems in her laboratory and sending 2 classical bits to Bob. For this, she simply measures her two qubits in the Bell basis and sends the outcome, i.e., one of the bitstrings $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$ to Bob. Bob uses this classical information to apply a quantum channel to his qubit and surprisingly he can be sure afterwards that it is in the state $\rho_{A'}$.

Let us analyze the protocol properly: We have to consider a quantum system consisting of three qubits with state space $\mathcal{H}_{A'} \otimes \mathcal{H}_A \otimes \mathcal{H}_B$. Initially, the system is in the quantum state

$$\sigma_{ini} = \rho_{A'} \otimes |\Omega_{AB}\rangle\langle\Omega_{AB}| = \frac{1}{2}\rho_{A'} \otimes \mathrm{vec}\left(\mathbb{1}_2\right)\mathrm{vec}\left(\mathbb{1}_2\right)^\dagger$$

Measuring the qubits $A'A$ using the Bell measurement, i.e., the PVM

$$\left\{\frac{1}{2}\,\mathrm{vec}\left(\sigma_{ij}\right)\mathrm{vec}\left(\sigma_{ij}\right)^\dagger \;:\; i,j \in \{0,1\}\right\}$$

leads to an outcome $(i,j) \in \{0,1\}^2$ and a post-measurement state

$$\sigma_{pm}^{(i,j)} = \frac{\left(\mathrm{vec}\left(\sigma_{ij}\right)^\dagger \otimes \mathbb{1}_{\mathcal{H}_B}\right)\sigma_{ini}\left(\mathrm{vec}\left(\sigma_{ij}\right) \otimes \mathbb{1}_{\mathcal{H}_B}\right)}{\mathrm{Tr}\left[\left(\mathrm{vec}\left(\sigma_{ij}\right)^\dagger \otimes \mathbb{1}_{\mathcal{H}_B}\right)\sigma_{ini}\left(\mathrm{vec}\left(\sigma_{ij}\right) \otimes \mathbb{1}_{\mathcal{H}_B}\right)\right]}.$$

The crucial step is easiest expressed in the following way. By using the necklace identity repeatedly, we find that

$$\begin{aligned}
&\left(\sqrt{\rho_{A'}} \otimes \mathrm{vec}\left(\mathbb{1}_2\right)^\dagger\right)\left(\mathrm{vec}\left(\sigma_{ij}\right) \otimes \mathbb{1}_{\mathcal{H}_B}\right) \\
&\quad= \left(\mathbb{1}_{\mathcal{H}_{A'}} \otimes \mathrm{vec}\left(\mathbb{1}_2\right)^\dagger\right)\left(\left(\sqrt{\rho_{A'}} \otimes \mathbb{1}_{\mathcal{H}_A}\right)\mathrm{vec}\left(\sigma_{ij}\right) \otimes \mathbb{1}_{\mathcal{H}_B}\right) \\
&\quad= \left(\mathbb{1}_{\mathcal{H}_{A'}} \otimes \mathrm{vec}\left(\mathbb{1}_2\right)^\dagger\right)\left(\mathrm{vec}\left(\sigma_{ij}\sqrt{\rho_{A'}}^T\right) \otimes \mathbb{1}_{\mathcal{H}_B}\right) \\
&\quad= \left(\mathbb{1}_{\mathcal{H}_{A'}} \otimes \mathrm{vec}\left(\mathbb{1}_2\right)^\dagger\right)\left(\left(\mathbb{1}_{\mathcal{H}_{A'}} \otimes \sigma_{ij}\sqrt{\rho_{A'}}^T\right)\mathrm{vec}\left(\mathbb{1}_2\right) \otimes \mathbb{1}_{\mathcal{H}_B}\right) \\
&\quad= \left(\mathbb{1}_{\mathcal{H}_{A'}} \otimes \mathrm{vec}\left(\sigma_{ij}^T\sqrt{\rho_{A'}}\right)^\dagger\right)\left(\mathrm{vec}\left(\mathbb{1}_2\right) \otimes \mathbb{1}_{\mathcal{H}_B}\right) \\
&\quad= \sqrt{\rho_{A'}}\sigma_{ij}^T.
\end{aligned}$$

We conclude that

$$\begin{aligned}
&\left(\mathrm{vec}\left(\sigma_{ij}\right)^\dagger \otimes \mathbb{1}_{\mathcal{H}_B}\right)\left(\rho_{A'} \otimes \mathrm{vec}\left(\mathbb{1}_2\right)\mathrm{vec}\left(\mathbb{1}_2\right)^\dagger\right)\left(\mathrm{vec}\left(\sigma_{ij}\right) \otimes \mathbb{1}_{\mathcal{H}_B}\right) \\
&\quad= \left(\mathrm{vec}\left(\sigma_{ij}\right) \otimes \mathbb{1}_{\mathcal{H}_B}\right)\left(\sqrt{\rho_{A'}} \otimes \mathrm{vec}\left(\mathbb{1}_2\right)\right)\left(\sqrt{\rho_{A'}} \otimes \mathrm{vec}\left(\mathbb{1}_2\right)^\dagger\right)\left(\mathrm{vec}\left(\sigma_{ij}\right) \otimes \mathbb{1}_{\mathcal{H}_B}\right) \\
&\quad= \sigma_{ij}^T\rho_{A'}\sigma_{ij}^T = \sigma_{ij}\rho_{A'}\sigma_{ij},
\end{aligned}$$

since $\sigma_{ij}^T = \pm\sigma_{ij}$. This shows that the post-measurment state of Bob's system is

$$\sigma_{pm}^{(i,j)} = \sigma_{ij}\rho_{A'}\sigma_{ij},$$

after Alice obtained the measurement outcome $(i,j)$. Note that each outcome $(i,j) \in \{0,1\}^2$ occurs with probability $1/4$. If Alice communicates her measurement outcome $(i,j)$ to Bob, he can apply the corresponding quantum channel $T_{ij} : B\left(\mathbb{C}^2\right) \to B\left(\mathbb{C}^2\right)$ given by $T_{ij}(X) = \sigma_{ij}X\sigma_{ij}$ to his system. After this, his qubit will be in the state $T_{ij}\left(\sigma_{pm}^{(i,j)}\right) = \rho_{A'}$.

As for superdense coding, we can express the teleporation protocol in terms of a resource inequality: We can use one ebit and two applications of a perfect classical bit channel to obtain one application of a perfect qubit channel. Or, to write it as a formula

$$[\text{ebit}(A \leftrightarrow B)] + 2\,[\text{cbit}(A \to B)] \geqslant [\text{qbit}(A \to B)]\,.$$