# UNIVERSITY OF OSLO

## Faculty of Mathematics and Natural Sciences

Examination in: MAT 4430 — Quantum information theory (Mock exam)

Day of examination: Whenever

Examination hours: T – T+4h

This problem set consists of 4 pages.

Appendices: None

Permitted aids: none.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

*Note: The exam consists of 10 subexercises which all give maximally 10 points. To get a point for a subexercise it is expected that you give an explanation of your solution.*

## Problem 1

Consider the linear map $T : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ given by

$$T\left(\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{3}x_1 + \frac{2}{3}x_4 & \frac{1}{6}x_2 \\ \frac{1}{6}x_3 & \frac{1}{3}x_4 + \frac{2}{3}x_1 \end{pmatrix}$$

### 1a

Show that $T$ is a quantum channel.

### 1b

Compute a Kraus decomposition for $T$.

### 1c

Compute a Stinespring dilation of $T$.

### 1d

Is the quantum channel $T$ entanglement breaking?

*(Continued on page 2.)*

# Problem 2

Consider a quantum channel $T : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ with Choi operator $C_T$ and the following scenario: Alice and Bob are two scientists who each posess a qubit quantum system labelled $A$ and $B$, respectively. The joint quantum state of these qubits is given by the normalized Choi operator $\rho_{AB} = C_T/\mathrm{Tr}\,[C_T]$.

## 2a

Assume that there is another qubit quantum system labelled $A'$ in Alice's laboratory. Initially, this quantum system is uncorrelated with the system $A$ and in a quantum state $\sigma_{A'}$ unknown to Alice. Compute the final quantum state $\tau_B$ of Bob's system after Alice measures her systems $A'A$ using the Bell measurement and obtained a particular outcome $(i, j) \in \{0, 1\}^2$.

## 2b

Assume that

$$T = p_0 \mathrm{id}_2 + p_1 \mathrm{Ad}_{\sigma_x} + p_2 \mathrm{Ad}_{\sigma_y} + p_3 \mathrm{Ad}_{\sigma_3},$$

for probabilities $p_0, p_1, p_2, p_3 \in [0, 1]$ summing to 1. After her measurement Alice communicates the measurement outcome $(i, j) \in \{0, 1\}^2$ to Bob. What quantum channel does Bob have to apply in order to be sure that his quantum system is in the quantum state $T(\sigma_{A'})$.

# Problem 3

In the following, let $\sigma_1, \sigma_2, \sigma_3 \in B(\mathbb{C}^2)$ denote the Pauli matrices. Let $P : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ denote a linear map of the form

$$P(x) = \mathrm{Tr}\,[x]\,\frac{\mathbb{1}_2}{2} + \frac{1}{2}\lambda_1 \mathrm{Tr}\,[\sigma_1 x]\,\sigma_1 + \frac{1}{2}\lambda_2 \mathrm{Tr}\,[\sigma_2 x]\,\sigma_2,$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$. Assume that $P$ is positive and show that there exists a $\lambda_3 \in [-1, 1]$ such that the map

$$T = P + \frac{1}{2}\lambda_3 \mathrm{Tr}\,[\sigma_3 x]\,\sigma_3,$$

is a quantum channel. Determine for which $\lambda_1$ and $\lambda_2$, the quantum channel $T$ can be entanglement breaking.

# Problem 4

Imagine you are a quantum telecommunication engineer. You have a device to produce a qubit in either of the states $\rho_0, \rho_1 \in D(\mathbb{C}^2)$ given by

$$\rho_0 = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \rho_1 = |+\rangle\langle +| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

*(Continued on page 3.)*

and send it to a distant location.

## 4a

Find a POVM $\mu : \{0,1\} \to B(\mathbb{C}^2)$ such that the classical channel $N_\mu : \{0,1\} \to \mathcal{P}(\{0,1\})$ with $N(x|y) = \mathrm{Tr}\,[\mu(x)\rho_y]$ is binary symmetric. Compute the capacity of the classical channel $N_\mu$ .

## 4b

Show that you can send classical information at a rate at least 0.3991 bits per use of your device.

## 4c

Argue that we can send classical information at a rate of at least 0.6 bits per use of the device, if we allow for global measurements at the receiving end.

*(Continued on page 4.)*

**Matlab-utskrift.**

```
>> A=[3/4 1/4
1/4 1/4]

A =

     3/4     1/4
     1/4     1/4


>> eig(A)

ans =

    0.8536
    0.1464

>> x=1/2 + 1/(2*sqrt(2))

x =

    0.8536

>> -x*log2(x)-(1-x)*log2(1-x)

ans =

    0.6009
```