EXERCISES 10

Excercise 1.

(1) Let $V : \mathcal{H}_A \to \mathcal{H}_A \otimes \mathcal{H}_E$ denote an isometry and $|\psi_E\rangle \in \mathcal{H}_E$ a normalized vector. Show that there exists a unitary $U : \mathcal{H}_A \otimes \mathcal{H}_E \to \mathcal{H}_A \otimes \mathcal{H}_E$ such that

$$U(\mathbb{1}_{\mathcal{H}_A} \otimes |\psi_E\rangle) = V.$$

(2) Show that for any quantum channel $T : B(\mathcal{H}) \to B(\mathcal{H})$ there exists a complex Euclidean space $\mathcal{H}_E$ such that for any normalized vector $|\psi_E\rangle \in \mathcal{H}_E$ there exists a unitary $U : \mathcal{H} \otimes \mathcal{H}_E \to \mathcal{H} \otimes \mathcal{H}_E$ such that

$$T(X) = \mathrm{Tr}_E \left[U(X \otimes |\psi_E\rangle\langle\psi_E|)U^\dagger \right].$$

for any $X \in B(\mathcal{H}_A)$. What is the minimal dimension of $\mathcal{H}_E$ needed to represent every quantum channel $T : B(\mathcal{H}) \to B(\mathcal{H})$ for a fixed complex Euclidean space $\mathcal{H}$.

Excercise 2 (The amplitude damping channel)**.** For $\gamma \in [0, 1]$ we define the *amplitude-damping channel* $N_\gamma : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ by

$$N_\gamma \left[\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \right] = \begin{pmatrix} x_{11} + \gamma x_{22} & \sqrt{1-\gamma}\, x_{12} \\ \sqrt{1-\gamma}\, x_{21} & (1-\gamma) x_{22} \end{pmatrix}.$$

(1) Compute the Choi matrix of N_γ. Show that N_γ is indeed a quantum channel.
(2) Compute a Choi-Kraus representation of N_γ.
(3) Compute a Stinespring dilation of N_γ.

Excercise 3 (Depolarizing channels)**.** For $p \in \mathbb{R}$ we define the map $D_p : B(\mathbb{C}^d) \to B(\mathbb{C}^d)$ by

$$D_p(X) = (1-p)\,\mathrm{Tr}\,[X]\,\frac{\mathbb{1}_d}{d} + pX,$$

for any $X \in B(\mathbb{C}^d)$.

(1) Compute the Choi operator of D_p. For which parameters $p \in \mathbb{R}$ is D_p a quantum channel?
(2) Compute the Choi-Kraus representation of D_p for the values of p where it is a quantum channel.
(3) Compute the Stinespring dilation of D_p for the values of p where it is a quantum channel.
(4) Compute a complementary channel D_p^c for dimension $d = 2$ of D_p for the values of p where it is a quantum channel.

Excercise 4 (Quantum teleportation)**.** Consider again the two researchers Alice and Bob who each have a laboratory. Alice's laboratory contains two quantum systems labelled 'A' and '$\tilde{A}$' and Bob's laboratory contains a quantum system labelled 'B'. Each of these quantum systems is a qubit. Initially, 'A' and 'B' are in a maximally entangled state. Assume furthermore, that the system '$\tilde{A}$' is initially in some quantum state $\rho_{\tilde{A}} \in D(\mathcal{H}_{\tilde{A}})$ unknown to Alice and Bob. Quantum teleportation is a protocol that lets Alice send this unknown state to Bob by performing a measurement and sending the measurement outcome to Bob. Proceed as follows:

(1) Recall the Bell basis $\{|\psi_{ij}\rangle\}_{ij} \subset \mathbb{C}^2 \otimes \mathbb{C}^2$ given by

$$|\psi_{ij}\rangle = \frac{1}{\sqrt{2}} \operatorname{vec}(\sigma_{ij}),$$

where

$$\sigma_{00} = \mathbb{1}_2, \quad \sigma_{01} = \sigma_x, \quad \sigma_{10} = \sigma_y, \quad \sigma_{11} = \sigma_z,$$

are the identity matrix and the Pauli matrices. Compute the probability of obtaining the outcome $(k, l) \in \{0, 1\}^2$ when measuring the subsystems '$\tilde{A}A$' of the quantum state

$$\sigma_{ini} = \rho_{\tilde{A}} \otimes \omega_{AB},$$

using the von-Neumann measurement $\{|\psi_{ij}\rangle\langle\psi_{ij}|\}_{i,j}$ defined via the Bell basis.

(2) What is the post-measurement state σ^{ij}_{post} after obtaining outcome (i, j) in the measurement of (1)?

(3) If Alice communicates the measurement outcome (i, j) of the Bell measurement to Bob, what quantum channel can he apply so that his system is in the final state $\rho_{\tilde{A}}$?

(4) How can we extend this protocol to quantum systems with dimension $d > 2$?

Excercise 5.

(1) Consider complex Euclidean spaces $\mathcal{H}$ and $\mathcal{H}'$ such that $\dim(\mathcal{H}) \leqslant \dim(\mathcal{H}')$. Let $\{|v_i\rangle\}_{i=1}^k \subset \mathcal{H}$ and $\{|w_i\rangle\}_{i=1}^k \subset \mathcal{H}'$ denote orthonormal sets of vectors. Show that there exists an isometry $V : \mathcal{H} \to \mathcal{H}'$ such that $V|v_i\rangle = |w_i\rangle$ for all $i \in \{1, \ldots, k\}$.

(2) Let $\mathcal{H}_A, \mathcal{H}_E$ and $\mathcal{H}_{E'}$ denote complex Euclidean spaces. Consider a purification $|\psi_{AE}\rangle\langle\psi_{AE}| \in D(\mathcal{H}_A \otimes \mathcal{H}_E)$ of a quantum state $\rho_A \in D(\mathcal{H}_A)$ and a quantum state $\sigma_{AE'} \in D(\mathcal{H}_A \otimes \mathcal{H}_{E'})$ such that $\operatorname{Tr}_{E'}[\sigma_{AE'}] = \rho_A$. Show that there exists a quantum channel $S : B(\mathcal{H}_E) \to B(\mathcal{H}_{E'})$ such that

$$(\mathrm{id}_A \otimes S)(|\psi_{AE}\rangle\langle\psi_{AE}|) = \sigma_{AE'}.$$

Hint: Start by reducing the problem to the case, where $\sigma_{AE'}$ is a pure state as well.