EXERCISES 11

Excercise 1 (The fully depolarizing channel)**.** Consider the fully depolarizing channel $\Delta : B(\mathbb{C}^d) \to B(\mathbb{C}^d)$ given by

$$\Delta(X) = \mathrm{Tr}\,[X]\,\frac{\mathbb{1}_{\mathbb{C}^d}}{d}.$$

(1) Show that there exist unitary operators $\{U_1, \ldots, U_{d^2}\} \subset \mathcal{U}\left(\mathbb{C}^d\right)$ such that

$$\Delta = \frac{1}{d^2}\sum_{i=1}^{d^2} \mathrm{Ad}_{U_i}\,.$$

(Hint: Use the Choi-Jamiolkowski isomorphism to figure out which sets of unitary operators work here. Then, use a previous exercise.)

(2) Consider a quantum channel $T : B(\mathbb{C}^{d_A}) \to B(\mathbb{C}^{d_B})$. Use the open-system representation of T to show that there exists a dimension $d_E \in \mathbb{N}$ and a pure quantum state $|\psi_{EB}\rangle\langle\psi_{EB}| \in D\left(\mathbb{C}^{d_E} \otimes \mathbb{C}^{d_B}\right)$ such that

$$\frac{\mathbb{1}_{\mathbb{C}^{d_A}}}{d_A} \otimes \frac{\mathbb{1}_{\mathbb{C}^{d_E}}}{d_E} \otimes T(X)$$
$$= \frac{1}{d_E^2 d_A^2}\sum_{i=1}^{d_E^2}\sum_{j=1}^{d_A^2} \mathrm{Ad}_{\left(U_i^A \otimes V_j^E \otimes \mathbb{1}_{d_B}\right)} \circ \mathrm{Ad}_W\left(X \otimes |\psi_{EB}\rangle\langle\psi_{EB}|\right),$$

for some unitaries $\{U_1^A, \ldots, U_{d_A^2}^A\} \subset \mathcal{U}\left(\mathbb{C}^{d_A}\right)$ and $\{V_1^E, \ldots, V_{d_E^2}^E\} \subset \mathcal{U}\left(\mathbb{C}^{d_E}\right)$, and some unitary $W \in \mathcal{U}\left(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_E} \otimes \mathbb{C}^{d_B}\right)$.

Excercise 2 (Diagonalizing positive unital and trace-preserving qubit maps)**.** We want to diagonalize positive linear maps $P : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ satisfying $P(\mathbb{1}_{\mathbb{C}^2}) = \mathbb{1}_{\mathbb{C}^2}$ and $\mathrm{Tr}\,[P(X)] = \mathrm{Tr}\,[X]$ for any $X \in B(\mathbb{C}^2)$. For this we recall (from a previous exercise) that there is a 1-to-1 correspondence between the quantum states $D(\mathbb{C}^2)$ and the Euclidean unit ball $B_1(0) \subset \mathbb{R}^3$ (the Bloch ball) given by

$$\rho = \frac{1}{2}\left(\mathbb{1}_{\mathbb{C}^2} + w \cdot \sigma\right) \in D(\mathbb{C}^2) \quad \longleftrightarrow \quad w \in B_1(0),$$

where we use the shorthand

$$w \cdot \sigma = \sum_{i=1}^{3} w_i \sigma_i,$$

with the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We will sometimes write $\sigma_0 = \mathbb{1}_{\mathbb{C}^2}$ and we recall that the real span of $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ is $B(\mathbb{C}^2)_{sa}$ and the complex span is $B(\mathbb{C}^2)$. Proceed as follows:

(1) Consider a positive map $P : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ satisfying $P(\mathbb{1}_{\mathbb{C}^2}) = \mathbb{1}_{\mathbb{C}^2}$ and $\mathrm{Tr}\,[P(X)] = \mathrm{Tr}\,[X]$ for any $X \in B(\mathbb{C}^2)$. Show that $M_{ij} = \frac{1}{2}\,\mathrm{Tr}\,[\sigma_j P(\sigma_i)] \in \mathbb{R}$ for every $i, j \in \{1, 2, 3\}$ and that

$$P\left(\frac{1}{2}\left(\mathbb{1}_{\mathbb{C}^2} + w \cdot \sigma\right)\right) = \frac{1}{2}\left(\mathbb{1}_{\mathbb{C}^2} + (Mw) \cdot \sigma\right),$$

for any $w \in \mathbb{R}^3$, where we introduced $M \in B(\mathbb{R}^3)$ with entries M_{ij}.

(2) Consider the unitary

$$U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}.$$

Find an $R \in \mathrm{SO}(3)$ such that

$$\mathrm{Ad}_U \left(\frac{1}{2} \left(\mathbb{1}_{\mathbb{C}^2} + w \cdot \sigma \right) \right) = \frac{1}{2} \left(\mathbb{1}_{\mathbb{C}^2} + (Rw) \cdot \sigma \right).$$

(3) Now, consider a general $R \in \mathrm{SO}(3)$, i.e., corresponding to a 3-dimensional rotation of the ball $B_1(0)$. Show that there exists a $U \in \mathcal{U}\left(\mathbb{C}^2\right)$ such that

$$\mathrm{Ad}_U \left(\frac{1}{2} \left(\mathbb{1}_{\mathbb{C}^2} + w \cdot \sigma \right) \right) = \frac{1}{2} \left(\mathbb{1}_{\mathbb{C}^2} + (Rw) \cdot \sigma \right).$$

Hint: Start with rotations around the coordinate axes. Then, use that a general $R \in \mathrm{SO}(3)$ can be written as the product three rotations around the coordinate axes with certain rotation angles (sometimes called *Euler angles*).

(4) Consider a positive map $P : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ satisfying $P(\mathbb{1}_{\mathbb{C}^2}) = \mathbb{1}_{\mathbb{C}^2}$ and $\mathrm{Tr}\left[P(X)\right] = \mathrm{Tr}\left[X\right]$ for any $X \in B(\mathbb{C}^2)$. Use a modified singular value decomposition to show that there are unitary matrices $U_1, U_2 \in \mathcal{U}\left(\mathbb{C}^2\right)$ satisfying

$$\left(\mathrm{Ad}_{U_1} \circ P \circ \mathrm{Ad}_{U_2}\right)(\sigma_i) = \lambda_i \sigma_i,$$

for any $i \in \{0, 1, 2, 3\}$ and some real numbers $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and $\lambda_0 = 1$.

Excercise 3 (Visualizing positive maps)**.** For $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$, consider the trace-preserving map $P_\lambda : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ such that $P_\lambda(\mathbb{1}_{\mathbb{C}^2}) = \mathbb{1}_{\mathbb{C}^2}$ and

$$P_\lambda\left(\sigma_i\right) = \lambda_i \sigma_i$$

for $i \in \{1, 2, 3\}$.

(1) For which $\lambda \in \mathbb{R}^3$ is P_λ positive? Visualize this set.

(2) For which $\lambda \in \mathbb{R}^3$ is P_λ completely positive? Visualize this set.

Hint: Compute the Choi-Jamiolkowski operator of P_λ. Use that $\sigma_i \otimes \sigma_i$ and $\sigma_j \otimes \sigma_j$ commute and are simultaneously diagonalized by the Bell basis.

(3) For which $\lambda \in \mathbb{R}^3$ is $P_\lambda = \vartheta \circ T$ for the transpose $\vartheta : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ and a completely positive map $T : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$? Visualize this set.

(4) Using your visualizations and Exercise 2, show that any positive map $P : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ satisfying $P(\mathbb{1}_{\mathbb{C}^2}) = \mathbb{1}_{\mathbb{C}^2}$ and $\mathrm{Tr}\left[P(X)\right] = \mathrm{Tr}\left[X\right]$ for any $X \in B(\mathbb{C}^2)$ can be written as

$$P = T_1 + \vartheta \circ T_2,$$

for completely positive maps $T_1, T_2 : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ and the transpose map $\vartheta : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$.

Excercise 4 (Entanglement manipulation)**.** Let $\mathcal{H}_{A_1}, \mathcal{H}_{A_2}, \mathcal{H}_{B_1}, \mathcal{H}_{B_2}$ denote complex Euclidean spaces. A quantum channel

$$T : B(\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1}) \to B(\mathcal{H}_{A_2} \otimes H_{\mathcal{B}_2})$$

is called a *separable*[1] *quantum channel* if it can be written as

$$T = \sum_{n=1}^{N} S_A^{(n)} \otimes R_B^{(n)},$$

[1]Do not confuse this with separable quantum states!

for completely positive maps $S_A^{(n)} : B(\mathcal{H}_{A_1}) \to B(\mathcal{H}_{A_2})$ and $R_B^{(n)} : B(\mathcal{H}_{B_1}) \to B(\mathcal{H}_{B_2})$. Let us denote the set of separable channels by

$$\mathrm{SepC}\left((\mathcal{H}_{A_1}, \mathcal{H}_{B_1}) \to (\mathcal{H}_{A_2}, \mathcal{H}_{B_2})\right).$$

(1) Show that $T(\rho_{AB}) \in \mathrm{Sep}(\mathcal{H}_{A_2}, \mathcal{H}_{B_2})$ for any $\rho_{AB} \in \mathrm{Sep}(\mathcal{H}_{A_1}, \mathcal{H}_{B_1})$ and any $\mathrm{SepC}((\mathcal{H}_{A_1}, \mathcal{H}_{B_1}) \to (\mathcal{H}_{A_2}, \mathcal{H}_{B_2}))$. The separable states are preserved under separable operations!

(2) Using the teleportation protocol, construct a separable quantum channel

$$T \in \mathrm{SepC}\left((\mathbb{C}^2 \otimes \mathbb{C}^2, \mathbb{C}^2) \to (\mathbb{C}^2 \otimes \mathbb{C}^2, \mathbb{C}^2)\right),$$

such that

$$T(\rho \otimes \omega_{AB}) = \frac{1}{4}(\mathbb{1}_{\mathbb{C}^2} \otimes \mathbb{1}_{\mathbb{C}^2}) \otimes \rho \in D(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2),$$

for any quantum state $\rho \in D(\mathbb{C}^2)$ and where $\omega_{AB} \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$ denotes the (normalized) maximally entangled state. How does this result generalize to dimensions $d > 2$.

(3) Show that any quantum state $\rho \in D(\mathbb{C}^d \otimes \mathbb{C}^d)$ can be written as

$$\rho = T(\omega_d),$$

for some separable quantum channel $T \in \mathrm{SepC}\left((\mathbb{C}^d, \mathbb{C}^d) \to (\mathbb{C}^d, \mathbb{C}^d)\right)$, and where $\omega_d \in D(\mathbb{C}^d \otimes \mathbb{C}^d)$ denotes the normalized maximally entangled state. This is why ω_d is called *maximally* entangled.