EXERCISES 12

Excercise 1 (Properties of the von Neumann entropy)**.** For a quantum state $\rho \in D(\mathcal{H})$ we define the *von Neumann entropy* as

$$H(\rho) = -\operatorname{Tr}\left[\rho \log(\rho)\right],$$

where the logarithm is in base 2.

(1) Show that $0 \leqslant H(\rho) \leqslant \log\left(\dim(\mathcal{H})\right)$. For which quantum states do we have $H(\rho) = 0$ and for which quantum states do we have $H(\rho) = \log\left(\dim(\mathcal{H})\right)$?
(2) Assume that $\rho = \operatorname{Tr}_B\left[|\psi_{AB}\rangle\langle\psi_{AB}|\right]$ and $\sigma = \operatorname{Tr}_A\left[|\psi_{AB}\rangle\langle\psi_{AB}|\right]$ for some pure state $|\psi_{AB}\rangle \in H_A \otimes H_B$. Show that $H(\rho) = H(\sigma)$.
(3) Consider a probability distribution $p \in \mathcal{P}\left(\{1, \ldots, N\}\right)$ and quantum states $\rho_1, \ldots, \rho_N \in D(\mathcal{H})$ satisfying $\rho_i \rho_j = 0$ for whenever $i \neq j$. Show that

$$H\left(\sum_{i=1}^{N} p_i \rho_i\right) = H(p) + \sum_{i=1}^{N} p_i H(\rho_i).$$

Excercise 2 (Some classical information theory)**.** For finite alphabets Σ_A and Σ_B let $N : \Sigma_A \to \mathcal{P}(\Sigma_B)$ denote a classical channel. Show that

$$C(N) \leqslant \min(\log(|\Sigma_A|), \log(|\Sigma_B|)).$$

Excercise 3 (The fidelity)**.** Let $\mathcal{H}$ denote a complex Euclidean space. For quantum states $\rho, \sigma \in D(\mathcal{H})$, we define the *fidelity* by

$$F(\rho, \sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1,$$

where $\sqrt{\cdot}$ denotes the unique positive square root and $\|\cdot\|_1$ is the trace norm (i.e., the matrix 1-norm). The fidelity measures the distance between the two quantum states, and in the following we will prove some of its properties.

(1) Show that

$$F(\rho, \sigma) = \operatorname{Tr}\left[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right],$$

for all quantum states $\rho, \sigma \in D(\mathcal{H})$.
(2) Show that $F(\rho, \sigma) = F(\sigma, \rho)$ for all quantum states $\rho, \sigma \in D(\mathcal{H})$.
(3) Show $F(\rho, \sigma) \geqslant 0$ with equality if and only if $\rho\sigma = 0$.
(4) Show that $F(\rho, \sigma) \leqslant 1$ with equality if and only if $\rho = \sigma$.
(5) Show that $F(V\rho V^\dagger, V\sigma V^\dagger) = F(\rho, \sigma)$ for any isometry $V : \mathcal{H} \to \mathcal{H}'$ into a complex Euclidean space $\mathcal{H}'$.
(6) Show that $F(|\psi\rangle\langle\psi|, \sigma) = \sqrt{\langle\psi|\sigma|\psi\rangle}$ for any pure quantum state $|\psi\rangle\langle\psi| \in B\left(\mathcal{H}\right)$ and any $\sigma \in D(\mathcal{H})$.

Excercise 4 (Entropies and measurements)**.** Consider a quantum state $\rho \in D(\mathcal{H})$ and a projection-valued measure $\mu : \Sigma \to B(\mathcal{H})^+$, i.e., a POVM such that the operators $\mu(x)$ are projections. Recall that measuring ρ using μ gives an outcome $x \in \Sigma$ with probability

$$p(x) = \operatorname{Tr}\left[\mu(x)\rho\right],$$

and after obtaining this outcome, the system will be in the quantum state

$$\rho_x = \frac{\mu(x)\rho\mu(x)}{\operatorname{Tr}\left[\mu(x)\rho\right]}.$$

If we do *not* learn the outcome $x \in \Sigma$ of the measurement, then the quantum system is in the quantum state

$$\rho_{meas} = \sum_{x \in \Sigma} p(x)\rho_x = \sum_{x \in \Sigma} \mu(x)\rho\mu(x).$$

(1) Show that

$$H(\rho_{meas}) \geqslant H(\rho),$$

i.e., a projective measurement increases the entropy if we do not learn the measurement outcome. (*Hint:* You may use the data-processing inequality for the relative entropy, but there is also a way of solving the exercise without it.)

(2) Find a 2-outcome instrument $\{T_1, T_2\}$ with completely positive maps $T_1, T_2 : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ and a quantum state $\rho \in D(\mathbb{C}^2)$ such that

$$H\left(T_1(\rho) + T_2(\rho)\right) < H(\rho).$$