EXERCISES 14

Excercise 1. Let $\sigma \in D(\mathcal{H})$ denote a quantum state and consider the quantum channel $T_p : B(\mathcal{H}) \to B(\mathcal{H})$ with a parameter $p \in [0,1]$ given by

$$T_p(X) = (1-p)X + p \operatorname{Tr}[X]\, \sigma.$$

Show that we have

$$H\left(T_p^{\otimes n}(\rho)\right) \geqslant (1-p)H(\rho) + pnH(\sigma),$$

for any $n \in \mathbb{N}$ and any $\rho \in D(\mathcal{H}^{\otimes n})$.

Hint: The case $\sigma = \mathbb{1}_{\mathcal{H}}/\dim(\mathcal{H})$ was done in the lectures.

Excercise 2. Use the open-system representation of quantum channels to show that

$$\|T(X)\|_1 \leqslant \|X\|_1, \tag{1}$$

for every quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ and every $X \in B(\mathcal{H}_A)$. In particular, this shows that for quantum states $\rho, \sigma \in D(\mathcal{H}_A)$ we have

$$\|T(\rho) - T(\sigma)\|_1 \leqslant \|\rho - \sigma\|_1.$$

Is (1) still true if we replace $\|\cdot\|_1$ by the Hilbert-Schmidt norm $\|\cdot\|_2$ or the operator norm $\|\cdot\|_\infty$?

Excercise 3. Consider a quantum state $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ and recall the quantum conditional entropy

$$H(A|B)_{\rho_{AB}} = H(\rho_{AB}) - H(\rho_B).$$

We aim to show that $H(A|B)_{\rho_{AB}} \geqslant 0$ whenever ρ_{AB} is separable.

(1) Show that

$$H(A|B)_{\rho_{AB}} \geqslant 0,$$

whenever

$$\rho_{AB} = \sum_{i=1}^{k} p_i \sigma_i^A \otimes |i\rangle\langle i|.$$

(2) Show that

$$H(A|B)_{\rho_{AB}} = H(\rho_A) - D(\rho_{AB} \| \rho_A \otimes \rho_B).$$

(3) Show that for any quantum channel $T : B(\mathcal{H}_B) \to B(\mathcal{H}_{B'})$ we have

$$H(A|B')_{(\mathrm{id}_A \otimes T)(\rho_{AB})} \geqslant H(A|B)_{\rho_{AB}}.$$

(4) Given a separable quantum state

$$\rho_{AB'} = \sum_{i=1}^{k} p_i \sigma_i^A \otimes \tau_i^{B'},$$

find a quantum channel $T : B(\mathcal{H}_B) \to B(\mathcal{H}_{B'})$ for $\mathcal{H}_B = \mathbb{C}^k$ such that

$$\rho_{AB'} = (\mathrm{id}_A \otimes T)\left(\sum_{i=1}^{k} p_i \sigma_i^A \otimes |i\rangle\langle i|\right).$$

Conclude that

$$H(A|B')_{\rho_{AB'}} \geqslant 0.$$

Excercise 4 (Typical projectors)**.** Consider a quantum state $\rho \in D(\mathcal{H})$ with spectral decomposition $\rho = \sum_{x=1}^{d} p(x)|v_x\rangle\langle v_x|$ where $p \in \mathcal{P}(\{1,\dots,d\})$ is a probability distribution and $|v_x\rangle \in \mathcal{H}$. Recall the ϵ-typical sequences $\mathcal{T}_{n,\epsilon}(p)$ from the first part of the course, and define a projector

$$\Pi_{n,\epsilon} = \sum_{(x_1,\dots,x_n)\in\mathcal{T}_{n,\epsilon}(p)} |v_{x_1}\rangle\langle v_{x_1}| \otimes \cdots \otimes |v_{x_n}\rangle\langle v_{x_n}| \in B\left(\mathcal{H}^{\otimes n}\right).$$

Show the following statements:

(1) For any $\epsilon > 0$ we have
$$\mathrm{Tr}\left[\Pi_{n,\epsilon}\rho^{\otimes n}\right] \to 1,$$
as $n \to \infty$.

(2) We have
$$\mathrm{Tr}\left[\Pi_{n,\epsilon}\right] < 2^{nH(\rho)+\epsilon}.$$

(3) For every $n \in \mathbb{N}$ let $\tilde{\Pi}_n \in B\left(\mathcal{H}^{\otimes n}\right)$ denote a projector such that $\mathrm{rank}\left(\tilde{\Pi}_n\right) \leqslant 2^{nR}$ for some $R < H(\rho)$. Show that
$$\mathrm{Tr}\left[\tilde{\Pi}_n\rho^{\otimes n}\right] \to 0,$$
as $n \to \infty$.