EXERCISES 4

Excercise 1 (Sums of projections)**.** For a complex Euclidean space $\mathcal{H}$ consider projections $\{P_n\}_{n=1}^N \subset B(\mathcal{H})$ such that

$$\sum_{n=1}^{N} P_n = \mathbb{1}_{\mathcal{H}}. \tag{1}$$

Show that $P_i P_j = 0$ all $i \neq j$. What is the maximal number $N \in \mathbb{N}$ such that there exists a set of projections $\{P_n\}_{n=1}^N \subset B(\mathcal{H})$ satisfying (1).

Excercise 2 (Positivity of certain matrices)**.** Let $\mathcal{H}$ denote a complex Euclidean space.

(1) For $x \in \mathbb{C}$ consider the operator $A \in B\left(\mathbb{C}^2\right)$ given by

$$A = \begin{pmatrix} 1 & \overline{x} \\ x & 1 \end{pmatrix}.$$

Show that A is positive semidefinite if and only if $|x| \leqslant 1$.

(2) For $X \in B(\mathcal{H})$ consider the operator $B \in B(\mathcal{H} \oplus \mathcal{H})$ given by the matrix

$$B = \begin{pmatrix} \mathbb{1}_{\mathcal{H}} & X^\dagger \\ X & \mathbb{1}_{\mathcal{H}} \end{pmatrix}.$$

Show that B is positive semidefinite if and only if the operator norm $\|X\| \leqslant 1$.

(3) Show that an operator $C \in B(\mathcal{H})$ is positive if and only if the operator

$$\begin{pmatrix} C & C \\ C & C \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{H})$$

is positive semidefinite.

Excercise 3 (The qubit)**.** Let us consider the complex Euclidean space $\mathcal{H} = \mathbb{C}^2$, and recall that the quantum states on $\mathcal{H}$ are the positive semidefinite matrices in $B(\mathcal{H})$ with trace equal to 1. We refer to this set as $D(\mathbb{C}^2)$. How can we understand this set geometrically?

(1) Consider the so-called Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that $\{\mathbb{1}_2, \sigma_1, \sigma_2, \sigma_3\}$ is an orthogonal basis of the Hilbert-Schmidt inner product space, i.e., the vector space $B(\mathcal{H})$ with the inner product $\langle A, B \rangle = \mathrm{Tr}\left[A^\dagger B\right]$. Show that their real span is $B(\mathcal{H})_{sa}$.

(2) Show that we have

$$\left(\sum_{i=1}^{3} x_i \sigma_i\right)^\dagger \left(\sum_{i=1}^{3} x_i \sigma_i\right) = \|x\|_2^2 \mathbb{1}_2,$$

for any $x \in \mathbb{R}^3$. Conclude that the matrix

$$\rho_x = \frac{1}{2}\mathbb{1}_2 + \frac{1}{2}\sum_{i=1}^{3} x_i \sigma_i,$$

is a quantum state, i.e., positive and of unit trace, for any vector $x \in \mathbb{R}^3$ satisfying $\|x\|_2 \leqslant 1$.

(3) Show that every quantum state $\rho \in D(\mathbb{C}^2)$ can be written as $\rho = \rho_x$ for some $x \in \mathbb{R}^3$. Therefore, we can identify $D(\mathbb{C}^2)$ with the unit ball in the 2-norm. What are the pure states in this picture? What quantum state is at the center of the ball?

Excercise 4 (SICs)**.** Find four normalized vectors $|v_1\rangle, |v_2\rangle, |v_3\rangle, |v_4\rangle \in \mathbb{C}^2$ such that the absolute values $|\langle v_i|v_j\rangle|$ are the same for all $i \neq j$. You can start by verifying that in this case

$$|\langle v_i|v_j\rangle|^2 = \frac{2\delta_{ij} + 1}{3},$$

for any $i, j \in \{1, \ldots, 4\}$.