EXERCISES 5

Excercise 1. Let $\mathcal{H}$ denote a complex Euclidean space of dimension $d = \dim(\mathcal{H})$. Consider the complex vector space $B(\mathcal{H})$ and the real vector space $B(\mathcal{H})_{sa}$, i.e., we consider the selfadjoint operators on $\mathcal{H}$ as an $\mathbb{R}$-vector space.

(1) Show that $\dim(B(\mathcal{H})) = \dim(\mathcal{H})^2$ and find an orthonormal basis of the space $B(\mathcal{H})$ with respect to the Hilbert-Schmidt inner product.
(2) Show that $\dim(B(\mathcal{H})_{sa}) = \dim(\mathcal{H})^2$ and find an orthonormal basis of the space $B(\mathcal{H})_{sa}$ with respect to the Hilbert-Schmidt inner product (turning $B(\mathcal{H})_{sa}$ into a real Euclidean space).

Excercise 2 (Two decomposition theorems)**.**
Let $\mathcal{H}$ denote a complex Euclidean space.

(1) Show that any selfadjoint operator $H \in B(\mathcal{H})_{sa}$ can be written as a difference $H = P - Q$ for some positive semidefinite operators $P, Q \in B(\mathcal{H})^+$ satisfying $PQ = 0$.
(2) Show that any operator $X \in B(\mathcal{H})$ can be written as $X = H_1 + iH_2$ for some selfadjoint operators $H_1, H_2 \in B(\mathcal{H})_{sa}$.
(3) Show that every operator $X \in B(\mathcal{H})$ can be written as a linear combination of positive semidefinite operators. In other words: The $\mathbb{C}$-span of $B(\mathcal{H})^+$ is $B(\mathcal{H})$.

Excercise 3 (The Weyl-Heisenberg system)**.**
The goal of this exercise is to construct the so-called *Weyl-Heisenberg system*, an orthogonal basis consisting of unitary operators for the Hilbert-Schmidt inner product space $B(\mathbb{C}^d)$. This system generalizes the Pauli matrices to higher dimensions.

(1) Consider the cyclic shift operator $S : \mathbb{C}^d \to \mathbb{C}^d$ given by
$$S|i\rangle = |i+1\rangle,$$
on the computational basis $\{|0\rangle, \ldots, |d-1\rangle\}$, and where addition is mod d. Show that the powers S^k for $k \in \{0, \ldots, d-1\}$ are orthogonal with respect to the Hilbert-Schmidt inner product $\langle X, Y\rangle_{HS} = \mathrm{Tr}\left[X^\dagger Y\right]$.
(2) For the dth root of unity $\omega_d = \exp(\frac{2\pi i}{d})$ consider the diagonal matrix
$$D = \begin{pmatrix} 1 & & & & \\ & \omega_d & & & \\ & & \omega_d^2 & & \\ & & & \ddots & \\ & & & & \omega_d^{d-1} \end{pmatrix} \in B(\mathbb{C}^d).$$
Show that the powers D^l for $l \in \{0, \ldots, d-1\}$ are orthogonal with respect to the Hilbert-Schmidt inner product.
(3) Finally, define unitary operators $U_{kl} = S^k D^l$ and show that they form an orthogonal basis for the Hilbert-Schmidt inner product space $B(\mathbb{C}^d)$. What operators do you get for $d = 2$?

Excercise 4. Alice and Bob are playing a quantum guessing game similar to the guessing games we have considered before. Alice has a quantum device with two buttons labeled "0" and "1". After pressing the button "0" the device emits a particle in quantum state $\rho_0 \in D(\mathcal{H})$, and after pressing the button "1" the device

emits a particle in quantum state $\rho_1 \in D(\mathcal{H})$. Bob catches the emitted particle and measures it using some POVM. Then, he tries to guess whether Alice pressed button 0 or button 1. Let us assume that Alice presses button "0" with probability 1/2 and button "1" with probability 1/2, then, what is the optimal success probability of Bob's guess?

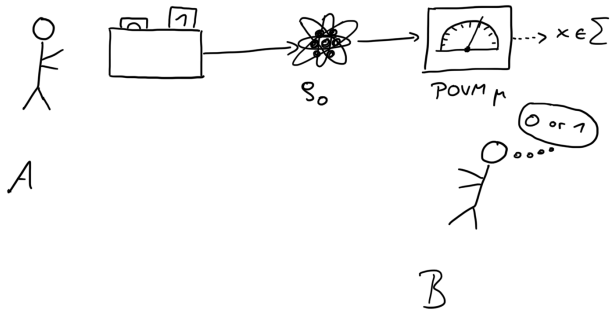


FIGURE 1. Alice and Bob at work.

(1) Let $\mu : \Sigma \to B(\mathcal{H})^+$ denote a POVM that Bob could use to measure the system. Show that after Bob has performed the measurement μ the optimal success probability of Bob guessing the button Alice has pushed is given by

$$p_{opt}(\mu) = \frac{1}{2} + \frac{1}{4} \sum_{x \in \Sigma} |\operatorname{Tr}[\mu(x)(\rho_0 - \rho_1)]|.$$

(2) Let p_{opt} denote the optimal probability of Bob guessing correctly, i.e., the solution to the optimization problem

$$p_{opt} = \sup\{p_{opt}(\mu) \ : \ \mu : \Sigma \to B(\mathcal{H})^+ \text{ POVM}\}.$$

Argue that the supremum is attained for a measurement with alphabet $\Sigma = \{1, \ldots, d^2\}$ with $d = \dim(\mathcal{H})$.

(3) Define the 1-norm of an operator $X \in B(\mathcal{H})$ as

$$\|X\|_1 = \operatorname{Tr}[|X|],$$

where $|X|$ denotes the positive square root of $X^\dagger X$. Show that the 1-norm $\|X\|_1$ coincides with the sum of singular values of X. Finally, find a measurement $\mu : \{1, \ldots, d\} \to B(\mathcal{H})^+$ such that

$$p_{opt}(\mu) = \frac{1}{2} + \frac{1}{4}\|\rho_0 - \rho_1\|_1.$$

Is this the optimal value p_{opt}?