EXERCISES 6

Excercise 1 (Identifying operators)**.** Show that for complex Euclidean spaces $\mathcal{H}_A, \mathcal{H}_B$ we have

$$B(\mathcal{H}_A) \otimes B(\mathcal{H}_B) \simeq B(\mathcal{H}_A \otimes \mathcal{H}_B).$$

Is it true that as vector spaces over $\mathbb{R}$ we have

$$B(\mathcal{H}_A)_{sa} \otimes B(\mathcal{H}_B)_{sa} \simeq B(\mathcal{H}_A \otimes \mathcal{H}_B)_{sa}, \tag{1}$$

for complex Euclidean spaces $\mathcal{H}_A, \mathcal{H}_B$? Is (1) true if $\mathcal{H}_A$ and $\mathcal{H}_B$ are real Euclidean spaces?

Excercise 2 (Vectorization)**.** Consider complex Euclidean spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ and let $d_A = \dim(\mathcal{H}_A)$ and $d_B = \dim(\mathcal{H}_B)$. Let us define the vectorization map $\mathrm{vec} : B(\mathcal{H}_A, \mathcal{H}_B) \to \mathcal{H}_A \otimes \mathcal{H}_B$ by

$$\mathrm{vec}(|v\rangle\langle w|) = |v\rangle \otimes \overline{|w\rangle},$$

and extended linearily.

(1) Show that $\mathrm{vec} : B(\mathcal{H}_A, \mathcal{H}_B) \to \mathcal{H}_A \otimes \mathcal{H}_B$ is an isometric isomorphism between the complex Euclidean spaces $B(\mathcal{H}_A, \mathcal{H}_B)$, i.e., the linear operators $X_{A\to B} : \mathcal{H}_A \to \mathcal{H}_B$ with the Hilbert-Schmidt inner product, and $\mathcal{H}_A \otimes \mathcal{H}_B$.

(2) Consider a vector $|v_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, and define an operator $X_{A\to B} : \mathcal{H}_A \to \mathcal{H}_B$ such that $|v_{AB}\rangle = \mathrm{vec}(X_{A\to B})$. Using the singular value decomposition of $X_{A\to B}$ show that there exist orthonormal bases $\{|a_i\rangle\}_i$ of $\mathcal{H}_A$ and $\{|b_i\rangle\}_i$ of $\mathcal{H}_B$, and positive numbers $\lambda_i \geqslant 0$ such that

$$|v_{AB}\rangle = \sum_{i=1}^{\min(d_A, d_B)} \lambda_i |a_i\rangle \otimes |b_i\rangle.$$

Excercise 3 (The partial transpose)**.** Recall the maximally entangled state

$$\omega_{\mathcal{H}} = \frac{1}{\dim(\mathcal{H})} |\Omega_{\mathcal{H}}\rangle\langle\Omega_{\mathcal{H}}|,$$

where

$$|\Omega_{\mathcal{H}}\rangle = \sum_{i=1}^{\dim(\mathcal{H})} |i\rangle \otimes |i\rangle.$$

(1) Consider the transpose map $\vartheta : B(\mathcal{H}) \to B(\mathcal{H})$ given by $\vartheta(X) = X^T$ in the computational basis. Let $P \in B(\mathcal{H})^+$ be positive semidefinite and show that $\vartheta(P) = P^T$ is positive semidefinite as well.

(2) Define the map $(\mathrm{id}_{\mathcal{H}} \otimes \vartheta) : B(\mathcal{H}) \otimes B(\mathcal{H}) \to B(\mathcal{H}) \otimes B(\mathcal{H})$ by setting $(\mathrm{id}_{\mathcal{H}} \otimes \vartheta)(X \otimes Y) = X \otimes Y^T$ and extended linearily. Compute $(\mathrm{id}_{\mathcal{H}} \otimes \vartheta)(\omega_{\mathcal{H}})$ and show that this operator is not positive semidefinite. Hint: Try it first in $\mathbb{C}^2$ and write down matrices.

(3) Show that $(\mathrm{id}_{\mathcal{H}} \otimes \vartheta)(\rho_{AB})$ is positive semidefinite whenever ρ_{AB} is separable. Conclude that $\omega_{\mathcal{H}}$ is entangled.

Excercise 4 (The realignment criterion)**.** For complex Euclidean spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ we define the realignment map $R : B(\mathcal{H}_A \otimes \mathcal{H}_B) \to B(\mathcal{H}_B \otimes \mathcal{H}_B, \mathcal{H}_A \otimes \mathcal{H}_A)$ by

$$R(|i_A\rangle\langle j_A| \otimes |k_B\rangle\langle l_B|) = |i_A\rangle\langle k_B| \otimes |j_A\rangle\langle l_B|,$$

and extended linearily.

(1) Show that
$$R(X \otimes Y) = |\operatorname{vec}(X^T)\rangle\langle\operatorname{vec}(Y^T)|,$$
for any $X \in B(\mathcal{H}_A)$ and $Y \in B(\mathcal{H}_B)$. Here, $\operatorname{vec}(\cdot)$ denotes the vectorization operation from exercise 2.

(2) Assume that $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ is a separable quantum state. Show that
$$\|R(\rho_{AB})\|_1 \leqslant 1.$$
Here, $\|\cdot\|_1$ denotes the 1-norm, i.e., the norm given by
$$\|A\|_1 = \operatorname{Tr}\left[\sqrt{A^\dagger A}\right],$$
for any $A \in B(\mathcal{H})$, which also equals the sum of singular values of A.

(3) Show that the maximally entangled state $\omega_{\mathcal{H}}$ satisfies
$$\|R(\omega_{\mathcal{H}})\|_1 > 1,$$
showing that $\omega_{\mathcal{H}}$ is entangled.