EXERCISES 7

Excercise 1. For a complex Euclidean space $\mathcal{H}$ consider the maximally entangled vector

$$|\Omega_{\mathcal{H}}\rangle = \sum_{i=1}^{\dim(\mathcal{H})} |i\rangle \otimes |i\rangle,$$

and the maximally entangled state

$$\omega_{\mathcal{H}} = \frac{1}{\dim(\mathcal{H})} |\Omega_{\mathcal{H}}\rangle\langle\Omega_{\mathcal{H}}|.$$

(1) Show that we have

$$(\mathbb{1}_{\mathcal{H}} \otimes X)|\Omega_{\mathcal{H}}\rangle = (X^T \otimes \mathbb{1}_{\mathcal{H}'})|\Omega_{\mathcal{H}'}\rangle$$

holds for any complex Euclidean space $\mathcal{H}'$ and any operator $X \in B(\mathcal{H}, \mathcal{H}')$, where the transpose is in the computational basis.

(2) Consider the so-called partial trace $(\mathrm{id}_{\mathcal{H}} \otimes \mathrm{Tr}) : B(\mathcal{H} \otimes \mathcal{H}) \to B(\mathcal{H})$ defined as $(\mathrm{id}_{\mathcal{H}} \otimes \mathrm{Tr})(X \otimes Y) = \mathrm{Tr}\,[Y]\, X$ for $X, Y \in B(\mathcal{H})$ and extended linearily. For an operator $Z \in B(\mathcal{H})$ we define the vector

$$|v_Z\rangle = (\mathbb{1}_{\mathcal{H}} \otimes Z)|\Omega_{\mathcal{H}}\rangle.$$

Show that

$$(\mathrm{id}_{\mathcal{H}} \otimes \mathrm{Tr})\left[|v_Z\rangle\langle v_Z|\right] = Z^T \overline{Z}.$$

(3) Show that for any quantum state $\rho \in D(\mathcal{H})$ there exists an operator $Z \in B(\mathcal{H})$ satisfying $\langle Z, Z\rangle_{HS} = 1$ such that

$$(\mathrm{id}_{\mathcal{H}} \otimes \mathrm{Tr})\left[|v_Z\rangle\langle v_Z|\right] = \rho.$$

Excercise 2 (Product measurements)**.** Consider a composite quantum system with a state space $\mathcal{H}_A \otimes \mathcal{H}_B$ for complex Euclidean spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. Let $\alpha : \Sigma_A \to B(\mathcal{H}_A)^+$ and $\beta : \Sigma_B \to B(\mathcal{H}_B)^+$ denote POVMs and recall that the POVM $\gamma : \Sigma_A \times \Sigma_B \to B(\mathcal{H}_A \otimes \mathcal{H}_B)^+$ corresponding to the simultaneous measurement of α on the A-subsystem and β on the B-subsystem is given by

$$\gamma(a, b) = \alpha(a) \otimes \beta(b), \tag{1}$$

for any $a \in \Sigma_A$ and any $b \in \Sigma_B$

(1) In the case where $\mathcal{H}_A = \mathcal{H}_B = \mathcal{H}$ we consider the maximally entangled state $\rho_{AB} = \omega_{\mathcal{H}}$ from the previous exercise. Show that

$$\mathrm{Tr}\left[\gamma(a, b)\omega_{\mathcal{H}}\right] = \frac{1}{\dim(\mathcal{H})} \langle \overline{\alpha(a)}, \beta(b)\rangle_{HS},$$

for any $a, b \in \Sigma$, and where the conjugation is in the computational basis.

(2) Show that the POVM $\gamma : \Sigma_A \times \Sigma_B \to B(\mathcal{H}_A \otimes \mathcal{H}_B)^+$ from above is the only POVM for which the formula

$$\mathrm{Tr}\left[\gamma(a, b)\rho_{AB}\right] = \mathrm{Tr}\left[\alpha(a)\sigma_A\right] \mathrm{Tr}\left[\beta(b)\tau_B\right].$$

holds for any product state $\rho_{AB} = \sigma_A \otimes \tau_B$. Argue why this formula makes sense physically if product states represent the quantum states of joint quantum systems whose components have never interacted.

Excercise 3. Assume that Alice and Bob perform a correlation experiment involving quantum systems with state spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ in a quantum state $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$. Alice is using POVMs $\alpha_1, \ldots, \alpha_n : \Sigma \to B(\mathcal{H}_A)^+$ and Bob is using POVMs $\beta_1, \ldots, \beta_n : \Sigma \to B(\mathcal{H}_B)^+$ giving rise to the correlation $(P(a,b|A_i,B_j))_{a,b,i,j}$ given by

$$P(a,b|i,j) = \mathrm{Tr}\left[(\alpha_i(a) \otimes \beta_j(b))\rho_{AB}\right].$$

(1) Show that this correlation is classical if ρ_{AB} is separable.
(2) Conclude that the maximally entangled state $\omega_{\mathbb{C}^2} \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is entangled.

Excercise 4. Let $n \in \mathbb{N}$ and Σ denote a finite alphabet. Assume that Alice and Bob perform some correlation experiment giving rise to a correlation $(P(a,b|A_i,B_j))_{a,b,i,j}$ with $a,b \in \Sigma$ and $i,j \in \{1,\ldots,n\}$. We call the correlation *non-signalling* if

$$\sum_{b\in\Sigma} P(a,b|A_i,B_j) = \sum_{b\in\Sigma} P(a,b|A_i,B_{j'})$$

for all $a \in \Sigma$ and any $i,j,j' \in \{1,\ldots,n\}$, and if

$$\sum_{a\in\Sigma} P(a,b|A_i,B_j) = \sum_{a\in\Sigma} P(a,b|A_{i'},B_j)$$

for all $b \in \Sigma$ and any $i,i',j \in \{1,\ldots,n\}$. The interpretation of these equations is that the probability distribution observed by Alice should not depend on Bob's choice of measurement, and the probability distribution observed by Bob should not depend on the choice of Alice's measurement.

(1) Show that quantum correlations are non-signalling.
(2) Show that the following correlation is not a quantum correlation.

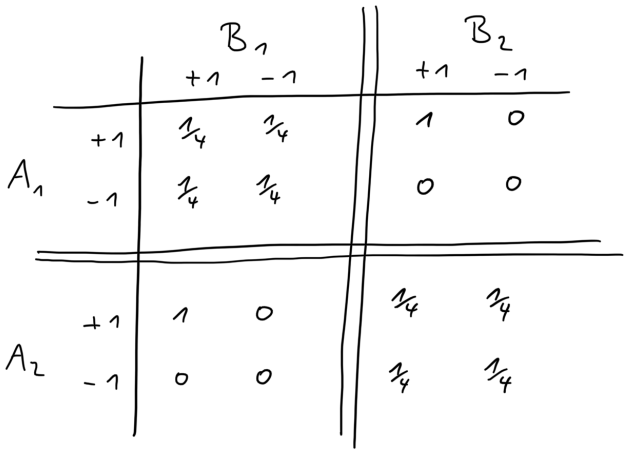


FIGURE 1. A correlation.

(3) (**Bonus question:**) Show that the following correlation is non-signalling and not a quantum correlation.

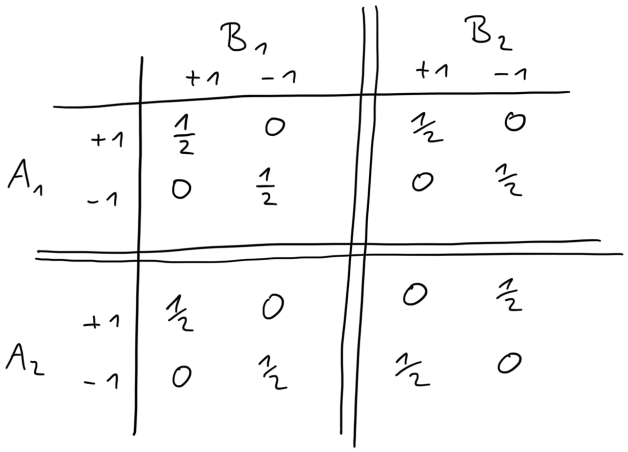


FIGURE 2. Yet another correlation.

Compare this correlation with the quantum correlation violating the CHSH inequality.