EXERCISES 10

Excercise 1. For a quantum state $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ we define the *mutual information* as

$$I(A:B)_{\rho_{AB}} = H(\rho_A) + H(\rho_B) - H(\rho_{AB}).$$

(1) Verify that

$$I(A:B)_{\rho_{AB}} = D(\rho_{AB} \| \rho_A \otimes \rho_B),$$

for any quantum state $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$.

(2) Using the data-processing inequality for the relative entropy, show that

$$I(A:B)_{\rho_{AB}} \leqslant I(A:(B,C))_{\rho_{ABC}},$$

for any quantum state $\rho_{ABC} \in D\left(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C\right)$.

Excercise 2. Consider a quantum channel $T : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ with Choi operator C_T and the following scenario: Alice and Bob are two scientists who each posess a qubit quantum system labelled A and B, respectively. The joint quantum state of these qubits is given by the normalized Choi operator $\rho_{AB} = C_T / \operatorname{Tr}[C_T]$.

(1) Assume that there is another qubit quantum system labelled A' in Alice's laboratory. Initially, this quantum system is uncorrelated with the system A and in a quantum state $\sigma_{A'}$ unknown to Alice. Compute the final quantum state τ_B of Bob's system after Alice measures her systems $A'A$ using the Bell measurement and obtained a particular outcome $(i,j) \in \{0,1\}^2$.

(2) Assume that

$$T = p_0 \mathrm{id}_2 + p_1 \operatorname{Ad}_{\sigma_x} + p_2 \operatorname{Ad}_{\sigma_y} + p_3 \operatorname{Ad}_{\sigma_3},$$

for probabilities $p_0, p_1, p_2, p_3 \in [0,1]$ summing to 1. After her measurement Alice communicates the measurement outcome $(i,j) \in \{0,1\}^2$ to Bob. What quantum channel does Bob have to apply in order to be sure that his quantum system is in the quantum state $T(\sigma_{A'})$.

Excercise 3 (Entropies and measurements)**.** Consider a quantum state $\rho \in D(\mathcal{H})$ and a projection-valued measure $\mu : \Sigma \to B(\mathcal{H})^+$, i.e., a POVM such that the operators $\mu(x)$ are projections. Recall that measuring ρ using μ gives an outcome $x \in \Sigma$ with probability

$$p(x) = \operatorname{Tr}[\mu(x)\rho],$$

and after obtaining this outcome, the system will be in the quantum state

$$\rho_x = \frac{\mu(x)\rho\mu(x)}{\operatorname{Tr}[\mu(x)\rho]}.$$

If we do *not* learn the outcome $x \in \Sigma$ of the measurement, then the quantum system is in the quantum state

$$\rho_{meas} = \sum_{x \in \Sigma} p(x)\rho_x = \sum_{x \in \Sigma} \mu(x)\rho\mu(x).$$

(1) Show that

$$H(\rho_{meas}) \geqslant H(\rho),$$

i.e., a projective measurement increases the entropy if we do not learn the measurement outcome. (*Hint:* You may use the data-processing inequality for the relative entropy, but there is also a way of solving the exercise without it.)

(2) Find a 2-outcome instrument $\{T_1, T_2\}$ with completely positive maps $T_1, T_2 : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ and a quantum state $\rho \in D(\mathbb{C}^2)$ such that

$$H\left(T_1(\rho) + T_2(\rho)\right) < H(\rho).$$

Excercise 4. Let $\sigma_1, \sigma_2, \sigma_3 \in B(\mathbb{C}^2)$ denote the Pauli matrices. Let $P : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ denote a linear map of the form

$$P(x) = \operatorname{Tr}[x]\,\frac{\mathbb{1}_2}{2} + \frac{1}{2}\lambda_1 \operatorname{Tr}[\sigma_1 x]\,\sigma_1 + \frac{1}{2}\lambda_2 \operatorname{Tr}[\sigma_2 x]\,\sigma_2,$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$. Assume that P is positive and show that there exists a $\lambda_3 \in [-1, 1]$ such that the map

$$T = P + \frac{1}{2}\lambda_3 \operatorname{Tr}[\sigma_3 x]\,\sigma_3,$$

is a quantum channel.