EXERCISES 3

Excercise 1 (The channel mutual information)**.** Compute the capacity $C(\mathrm{Erase}_p)$ for the binary erasure channel $\mathrm{Erase}_p : \{0,1\} \to \mathcal{P}(\{0,1\})$ introduced in the lecture.

Excercise 2 (Sums of projections)**.** For a complex Euclidean space $\mathcal{H}$ consider projections $\{P_n\}_{n=1}^N \subset B(\mathcal{H})$ such that

$$\sum_{n=1}^{N} P_n = \mathbb{1}_{\mathcal{H}}. \tag{1}$$

Show that $P_i P_j = 0$ all $i \neq j$. What is the maximal number $N \in \mathbb{N}$ such that there exists a set of projections $\{P_n\}_{n=1}^N \subset B(\mathcal{H})$ satisfying (1).

Excercise 3 (Positivity of certain matrices)**.** Let $\mathcal{H}$ denote a complex Euclidean space.

(1) For $x \in \mathbb{C}$ consider the operator $A \in B(\mathbb{C}^2)$ given by

$$A = \begin{pmatrix} 1 & \overline{x} \\ x & 1 \end{pmatrix}.$$

Show that A is positive semidefinite if and only if $|x| \leqslant 1$.

(2) For $X \in B(\mathcal{H})$ consider the operator $B \in B(\mathcal{H} \oplus \mathcal{H})$ given by the matrix

$$B = \begin{pmatrix} \mathbb{1}_{\mathcal{H}} & X^\dagger \\ X & \mathbb{1}_{\mathcal{H}} \end{pmatrix}.$$

Show that B is positive semidefinite if and only if the operator norm $\|X\| \leqslant 1$.

(3) Show that an operator $C \in B(\mathcal{H})$ is positive if and only if the operator

$$\begin{pmatrix} C & C \\ C & C \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{H})$$

is positive semidefinite.

Excercise 4 (Two decomposition theorems)**.**
Let $\mathcal{H}$ denote a complex Euclidean space.

(1) Show that any selfadjoint operator $H \in B(\mathcal{H})_{sa}$ can be written as a difference $H = P - Q$ for some positive semidefinite operators $P, Q \in B(\mathcal{H})^+$ satisfying $PQ = 0$.

(2) Show that any operator $X \in B(\mathcal{H})$ can be written as $X = H_1 + iH_2$ for some selfadjoint operators $H_1, H_2 \in B(\mathcal{H})_{sa}$.

(3) Show that every operator $X \in B(\mathcal{H})$ can be written as a linear combination of positive semidefinite operators. In other words: The $\mathbb{C}$-span of $B(\mathcal{H})^+$ is $B(\mathcal{H})$.