EXERCISES 4

Excercise 1 (The flip operator)**.** Find a matrix representation (in computational basis) of the linear operator $\mathbb{F} : \mathbb{C}^d \otimes \mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^d$ such that

$$\mathbb{F}(|a\rangle \otimes |b\rangle) = |b\rangle \otimes |a\rangle,$$

for every $|a\rangle, |b\rangle \in \mathbb{C}^d$. For convenience you may identify $\mathbb{C}^d \otimes \mathbb{C}^d \simeq \mathbb{C}^{d^2}$ using the lexicographic ordering of the computational basis on $\mathbb{C}^d \otimes \mathbb{C}^d$, i.e., such that

$$\begin{aligned}
|\mathbf{1}_2\rangle &= |1\rangle \otimes |1\rangle \\
|\mathbf{2}_2\rangle &= |1\rangle \otimes |2\rangle \\
|\mathbf{3}_2\rangle &= |2\rangle \otimes |1\rangle \\
|\mathbf{4}_2\rangle &= |2\rangle \otimes |2\rangle,
\end{aligned}$$

in the case $d = 2$.

Excercise 2 (Identifying operators)**.** Show that for complex Euclidean spaces $\mathcal{H}_A, \mathcal{H}_B$ we have

$$B(\mathcal{H}_A) \otimes B(\mathcal{H}_B) \simeq B(\mathcal{H}_A \otimes \mathcal{H}_B).$$

Is it true that as vector spaces over $\mathbb{R}$ we have

$$B(\mathcal{H}_A)_{sa} \otimes B(\mathcal{H}_B)_{sa} \simeq B(\mathcal{H}_A \otimes \mathcal{H}_B)_{sa}, \tag{1}$$

for complex Euclidean spaces $\mathcal{H}_A, \mathcal{H}_B$? Is (1) true if $\mathcal{H}_A$ and $\mathcal{H}_B$ are real Euclidean spaces?

Excercise 3 (The qubit)**.** Let us consider the complex Euclidean space $\mathcal{H} = \mathbb{C}^2$, and recall that the quantum states on $\mathcal{H}$ are the positive semidefinite matrices in $B(\mathcal{H})$ with trace equal to 1. We refer to this set as $D(\mathbb{C}^2)$. How can we understand this set geometrically?

(1) Consider the so-called Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that $\{\mathbb{1}_2, \sigma_1, \sigma_2, \sigma_3\}$ is an orthogonal basis of the Hilbert-Schmidt inner product space, i.e., the vector space $B(\mathcal{H})$ with the inner product $\langle A, B\rangle = \mathrm{Tr}\left[A^\dagger B\right]$. Show that their real span is $B(\mathcal{H})_{sa}$.

(2) Show that we have

$$\left(\sum_{i=1}^3 x_i \sigma_i\right)^\dagger \left(\sum_{i=1}^3 x_i \sigma_i\right) = \|x\|_2^2 \mathbb{1}_2,$$

for any $x \in \mathbb{R}^3$. Conclude that the matrix

$$\rho_x = \frac{1}{2}\mathbb{1}_2 + \frac{1}{2}\sum_{i=1}^3 x_i \sigma_i,$$

is a quantum state, i.e., positive and of unit trace, for any vector $x \in \mathbb{R}^3$ satisfying $\|x\|_2 \leqslant 1$.

(3) Show that every quantum state $\rho \in D(\mathbb{C}^2)$ can be written as $\rho = \rho_x$ for some $x \in \mathbb{R}^3$. Therefore, we can identify $D(\mathbb{C}^2)$ with the unit ball in the 2-norm. What are the pure states in this picture? What quantum state is at the center of the ball?

Excercise 4 (SICs)**.** Find four normalized vectors $|v_1\rangle, |v_2\rangle, |v_3\rangle, |v_4\rangle \in \mathbb{C}^2$ such that the absolute values $|\langle v_i|v_j\rangle|$ are the same for all $i \neq j$. You can start by verifying that in this case

$$|\langle v_i|v_j\rangle|^2 = \frac{2\delta_{ij}+1}{3},$$

for any $i, j \in \{1, \ldots, 4\}$.