EXERCISES 5

Excercise 1 (The Weyl-Heisenberg system)**.**
The goal of this exercise is to construct the so-called *Weyl-Heisenberg system*, an orthogonal basis consisting of unitary operators for the Hilbert-Schmidt inner product space $B(\mathbb{C}^d)$. This system generalizes the Pauli matrices to higher dimensions.

(1) Consider the cyclic shift operator $S : \mathbb{C}^d \to \mathbb{C}^d$ given by

$$S|i\rangle = |i+1\rangle,$$

on the computational basis $\{|0\rangle, \ldots, |d-1\rangle\}$, and where addition is mod d. Show that the powers S^k for $k \in \{0, \ldots, d-1\}$ are orthogonal with respect to the Hilbert-Schmidt inner product $\langle X, Y\rangle_{HS} = \mathrm{Tr}\left[X^\dagger Y\right]$.

(2) For the dth root of unity $\omega_d = \exp(\frac{2\pi i}{d})$ consider the diagonal matrix

$$D = \begin{pmatrix} 1 & & & & \\ & \omega_d & & & \\ & & \omega_d^2 & & \\ & & & \ddots & \\ & & & & \omega_d^{d-1} \end{pmatrix} \in B(\mathbb{C}^d).$$

Show that the powers D^l for $l \in \{0, \ldots, d-1\}$ are orthogonal with respect to the Hilbert-Schmidt inner product.

(3) Finally, define unitary operators $U_{kl} = S^k D^l$ and show that they form an orthogonal basis for the Hilbert-Schmidt inner product space $B(\mathbb{C}^d)$. What operators do you get for $d = 2$?

Excercise 2. For a complex Euclidean space $\mathcal{H}$ consider the maximally entangled vector

$$|\Omega_{\mathcal{H}}\rangle = \sum_{i=1}^{\dim(\mathcal{H})} |i\rangle \otimes |i\rangle,$$

and the maximally entangled state

$$\omega_{\mathcal{H}} = \frac{1}{\dim(\mathcal{H})} |\Omega_{\mathcal{H}}\rangle\langle\Omega_{\mathcal{H}}|.$$

(1) Show that we have

$$(\mathbb{1}_{\mathcal{H}} \otimes X)|\Omega_{\mathcal{H}}\rangle = (X^T \otimes \mathbb{1}_{\mathcal{H}'})|\Omega_{\mathcal{H}'}\rangle$$

holds for any complex Euclidean space $\mathcal{H}'$ and any operator $X \in B(\mathcal{H}, \mathcal{H}')$, where the transpose is in the computational basis.

(2) Consider the so-called partial trace $(\mathrm{id}_{\mathcal{H}} \otimes \mathrm{Tr}) : B(\mathcal{H} \otimes \mathcal{H}) \to B(\mathcal{H})$ defined as $(\mathrm{id}_{\mathcal{H}} \otimes \mathrm{Tr})(X \otimes Y) = \mathrm{Tr}\,[Y]\, X$ for $X, Y \in B(\mathcal{H})$ and extended linearily. For an operator $Z \in B(\mathcal{H})$ we define the vector

$$|v_Z\rangle = (\mathbb{1}_{\mathcal{H}} \otimes Z)|\Omega_{\mathcal{H}}\rangle.$$

Show that

$$(\mathrm{id}_{\mathcal{H}} \otimes \mathrm{Tr})\left[|v_Z\rangle\langle v_Z|\right] = Z^T \overline{Z}.$$

(3) Show that for any quantum state $\rho \in D(\mathcal{H})$ there exists an operator $Z \in B(\mathcal{H})$ satisfying $\langle Z, Z\rangle_{HS} = 1$ such that

$$(\mathrm{id}_{\mathcal{H}} \otimes \mathrm{Tr})\left[|v_Z\rangle\langle v_Z|\right] = \rho.$$

Excercise 3. Recall the different sets of correlations introduced in the lecture for bipartite correlation experiments.

(1) Show that classical correlations are quantum correlations.
(2) Show that quantum correlations are non-signalling.
(3) Show that the following correlation is not a quantum correlation.

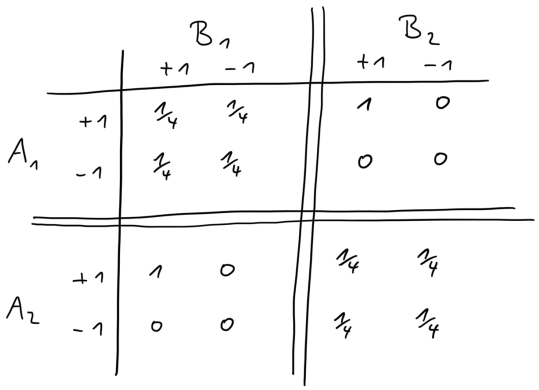


FIGURE 1. A correlation.

(4) (**Bonus question:**) Show that the following correlation is non-signalling and not a quantum correlation.

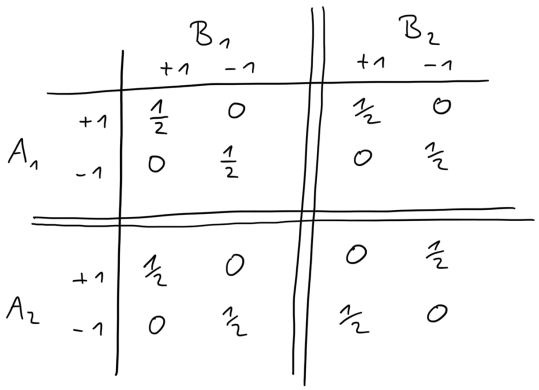


FIGURE 2. Yet another correlation.

Compare this correlation with the quantum correlation violating the CHSH inequality.

Excercise 4 (The realignment criterion)**.** For complex Euclidean spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ we define the realignment map $R : B(\mathcal{H}_A \otimes \mathcal{H}_B) \to B(\mathcal{H}_B \otimes \mathcal{H}_B, \mathcal{H}_A \otimes \mathcal{H}_A)$ by

$$R(|i_A\rangle\langle j_A| \otimes |k_B\rangle\langle l_B|) = |i_A\rangle\langle k_B| \otimes |j_A\rangle\langle l_B|,$$

and extended linearily.

(1) Show that
$$R(X \otimes Y) = |\operatorname{vec}(X^T)\rangle\langle\operatorname{vec}(Y^T)|,$$
for any $X \in B(\mathcal{H}_A)$ and $Y \in B(\mathcal{H}_B)$. Here, $\operatorname{vec}(\cdot)$ denotes the vectorization operation from exercise 2.

(2) Assume that $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ is a separable quantum state. Show that
$$\|R(\rho_{AB})\|_1 \leqslant 1.$$
Here, $\|\cdot\|_1$ denotes the 1-norm, i.e., the norm given by
$$\|A\|_1 = \operatorname{Tr}\left[\sqrt{A^\dagger A}\right],$$
for any $A \in B(\mathcal{H})$, which also equals the sum of singular values of A.

(3) Show that the maximally entangled state $\omega_{\mathcal{H}}$ satisfies
$$\|R(\omega_{\mathcal{H}})\|_1 > 1,$$
showing that $\omega_{\mathcal{H}}$ is entangled.