EXERCISES 6

Excercise 1 (Monogamy of entanglement)**.** Show that all purifications of a pure state $\rho_A = |\psi\rangle\langle\psi| \in D(\mathcal{H}_A)$ are of the form

$$\rho_{AE} = |\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|.$$

Now consider a composite quantum system 'AB' with tensor product state space $\mathcal{H}_A \otimes \mathcal{H}_B$ and assume that it is in a pure entangled quantum state $|\psi_{AB}\rangle\langle\psi_{AB}| \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$. Argue that there is no third quantum system that is entangled with the composite system 'AB'.

Excercise 2 (Examples of quantum channels)**.** Consider an isometry $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$. Show that the map $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ given by

$$T(X) = \mathrm{Tr}_E \left[VXV^\dagger \right],$$

is a quantum channel, i.e., it is completely positive and trace-preserving.

Excercise 3 (Purifications and reduced density matrices)**.** Consider a pure quantum state $|\psi_{AB}\rangle\langle\psi_{AB}| \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ and denote by ρ_A and ρ_B its two marginals, i.e.,

$$\rho_A = \mathrm{Tr}_B \left[|\psi_{AB}\rangle\langle\psi_{AB}| \right] \quad \text{, and} \quad \rho_B = \mathrm{Tr}_A \left[|\psi_{AB}\rangle\langle\psi_{AB}| \right].$$

Show the following statements:

(1) The spectra of ρ_A and ρ_B are the same. What is the relationship between the eigenvalues of these operators and the Schmidt coefficients of $|\psi_{AB}\rangle$?
(2) There exists an operator $X \in B(\mathcal{H}_A, \mathcal{H}_B)$ such that

$$\rho_A = X^T \overline{X} \text{ and } \rho_B = XX^\dagger.$$

Excercise 4. Consider a quantum state $\rho_A \in D(\mathcal{H}_A)$ with a purification $|\psi_{EA}\rangle\langle\psi_{EA}| \in D(\mathcal{H}_E \otimes \mathcal{H}_A)$ for some complex Euclidean space $\mathcal{H}_E$. Show that there is a complex Euclidean space $\mathcal{H}_{E'}$ with $\dim(\mathcal{H}_{E'}) = \mathrm{rk}\,(\rho_A)$ and a purification $|\phi_{E'A}\rangle\langle\phi_{E'A}| \in D(\mathcal{H}_{E'} \otimes \mathcal{H}_A)$ of ρ_A such that

$$|\psi_{EA}\rangle = (V \otimes \mathbb{1}_{\mathcal{H}_A})|\phi_{E'A}\rangle,$$

for some isometry $V : \mathcal{H}_{E'} \to \mathcal{H}_E$.