EXERCISES 7

Excercise 1 (The amplitude damping channel)**.** For $\gamma \in [0,1]$ we define the *amplitude-damping channel* $N_\gamma : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ by

$$N_\gamma \left[\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \right] = \begin{pmatrix} x_{11} + \gamma x_{22} & \sqrt{1-\gamma} x_{12} \\ \sqrt{1-\gamma} x_{21} & (1-\gamma) x_{22} \end{pmatrix}.$$

(1) Compute the Choi matrix of N_γ. Show that N_γ is indeed a quantum channel.
(2) Compute a Choi-Kraus representation of N_γ.
(3) Compute a Stinespring dilation of N_γ.

Excercise 2.

(1) Consider complex Euclidean spaces $\mathcal{H}$ and $\mathcal{H}'$ such that $\dim(\mathcal{H}) \leqslant \dim(\mathcal{H}')$. Let $\{|v_i\rangle\}_{i=1}^k \subset \mathcal{H}$ and $\{|w_i\rangle\}_{i=1}^k \subset \mathcal{H}'$ denote orthonormal sets of vectors. Show that there exists an isometry $V : \mathcal{H} \to \mathcal{H}'$ such that $V|v_i\rangle = |w_i\rangle$ for all $i \in \{1, \ldots, k\}$.
(2) Let $\mathcal{H}_A, \mathcal{H}_E$ and $\mathcal{H}_{E'}$ denote complex Euclidean spaces. Consider a purification $|\psi_{AE}\rangle\langle\psi_{AE}| \in D(\mathcal{H}_A \otimes \mathcal{H}_E)$ of a quantum state $\rho_A \in D(\mathcal{H}_A)$ and a quantum state $\sigma_{AE'} \in D(\mathcal{H}_A \otimes \mathcal{H}_{E'})$ such that $\mathrm{Tr}_{E'}[\sigma_{AE'}] = \rho_A$. Show that there exists a quantum channel $S : B(\mathcal{H}_E) \to B(\mathcal{H}_{E'})$ such that

$$(\mathrm{id}_A \otimes S)(|\psi_{AE}\rangle\langle\psi_{AE}|) = \sigma_{AE'}.$$

Hint: Start by reducing the problem to the case, where $\sigma_{AE'}$ is a pure state as well.

Excercise 3. Go through the material on quantum theory we have covered so far and think of a *good* question. Submit your question via email.

Excercise 4 (Ensemble representation theorem)**.** For a complex Euclidean space of dimension $\dim(\mathcal{H}) = d$ consider two sets $\{|\psi_n\rangle\}_{n=1}^N \subset \mathcal{H}$ and $\{|\phi_k\rangle\}_{k=1}^K \subset \mathcal{H}$ of vectors. We aim to show that

$$\sum_{n=1}^N |\psi_n\rangle\langle\psi_n| = \sum_{k=1}^K |\phi_k\rangle\langle\phi_k|.$$

if and only if there exists a unitary operator $U \in \mathcal{U}\left(\mathbb{C}^N\right)$ such that

$$|\psi_n\rangle = \sum_{k=1}^K U_{nk} |\phi_k\rangle,$$

where the smaller set of vectors is extended by zero vectors. To show this result apply the following steps, or find your own proof:

(1) Show the easy direction.
(2) Consider a set of orthonormal vectors $\{|e_k\rangle\}_{k=1}^K \subset \mathcal{H}$ and let $\{|\psi_n\rangle\}_{n=1}^N \subset \mathcal{H}$ for $N \geqslant K$ satisfy

$$\sum_{n=1}^N |\psi_n\rangle\langle\psi_n| = \sum_{k=1}^K |e_k\rangle\langle e_k|.$$

Show that

$$\{\left(|e_k\rangle\right\}_{k=1}^K)^\perp \subseteq \left(\{|\psi_n\rangle\}_{n=1}^N\right)^\perp,$$

where $\mathcal{S}^\perp$ denotes the orthogonal complement of a set $\mathcal{S} \subseteq \mathcal{H}$. Then, construct a unitary $U \in \mathcal{U}\left(\mathbb{C}^N\right)$ such that

$$\sum_{n=1}^{N} U_{kn}|\psi_n\rangle = |e_k\rangle,$$

and

$$\sum_{k=1}^{K} \overline{U_{kn}}|e_k\rangle = |\psi_n\rangle,$$

for each $n \in \{1, \ldots, N\}$ and $k \in \{1, \ldots, K\}$.

(3) Consider next the case, where $\{|e_k\rangle\}_{k=1}^{K} \subset \mathcal{H}$ are the eigenvectors corresponding to non-zero eigenvalues $\{\lambda_1, \ldots, \lambda_K\}$ of the operator

$$A = \sum_{n=1}^{N} |\psi_n\rangle\langle\psi_n|.$$

Construct a unitary operator $U \in \mathcal{U}\left(\mathbb{C}^N\right)$ such that

$$\sum_{n=1}^{N} U_{kn}|\psi_n\rangle = \sqrt{\lambda_k}|e_k\rangle,$$

and

$$\sum_{k=1}^{K} \overline{U_{kn}}\sqrt{\lambda_k}|e_k\rangle = |\psi_n\rangle,$$

for each $n \in \{1, \ldots, N\}$ and $k \in \{1, \ldots, K\}$.

(4) Finally, combine the insights of the previous points to prove the theorem stated in the beginning of this exercise.

(5) Relate this result to the Kraus representation of a completely positive map and show the theorem from the lecture notes.