EXERCISES 8

Excercise 1. Consider the linear map $T : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ given by

$$T\left(\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{3}x_1 + \frac{2}{3}x_4 & -\frac{1}{3}x_3 \\ -\frac{1}{3}x_2 & \frac{1}{3}x_4 + \frac{2}{3}x_1 \end{pmatrix}$$

Note the off-diagonal entries!

(1) Show that T is a quantum channel.
(2) Compute a Kraus decomposition for T.
(3) Compute a Stinespring dilation of T.

Excercise 2 (The swap test)**.** Let $d \in \mathbb{N}$ and consider the operators

$$M_0 = \frac{1}{2}\left(\mathbb{1}_d \otimes \mathbb{1}_d + \mathbb{F}\right) \quad \text{and} \quad M_1 = \frac{1}{2}\left(\mathbb{1}_d \otimes \mathbb{1}_d - \mathbb{F}\right).$$

Here, $\mathbb{F} \in B(\mathbb{C}^d \otimes \mathbb{C}^d)$ denotes the flip operator acting as $\mathbb{F}(|a\rangle \otimes |b\rangle) = |b\rangle \otimes |a\rangle$ on product states.

(1) Show that $\mu : \{0, 1\} \to B(\mathbb{C}^d \otimes \mathbb{C}^d)$ with $\mu(0) = M_0$ and $\mu(1) = M_1$ defines a POVM.
(2) Compute the probability of obtaining the outcomes 0 and 1 when measuring a pure product state $|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|$ using this POVM. Explain how this POVM can be used to determine whether two pure quantum states $|\psi\rangle\langle\psi|$ and $|\phi\rangle\langle\phi|$ produced by a quantum device are equal.

Excercise 3 (The fully depolarizing channel)**.** Consider the fully depolarizing channel $\Delta : B(\mathbb{C}^d) \to B(\mathbb{C}^d)$ given by

$$\Delta(X) = \mathrm{Tr}\left[X\right] \frac{\mathbb{1}_{\mathbb{C}^d}}{d}.$$

(1) Show that there exist unitary operators $\{U_1, \ldots, U_{d^2}\} \subset \mathcal{U}\left(\mathbb{C}^d\right)$ such that

$$\Delta = \frac{1}{d^2} \sum_{i=1}^{d^2} \mathrm{Ad}_{U_i}.$$

(Hint: Use the Choi-Jamiolkowski isomorphism to figure out which sets of unitary operators work here. Then, use a previous exercise.)

(2) Consider a quantum channel $T : B(\mathbb{C}^{d_A}) \to B(\mathbb{C}^{d_B})$. Use the open-system representation of T to show that there exists a dimension $d_E \in \mathbb{N}$ and a pure quantum state $|\psi_{EB}\rangle\langle\psi_{EB}| \in D\left(\mathbb{C}^{d_E} \otimes \mathbb{C}^{d_B}\right)$ such that

$$\frac{\mathbb{1}_{\mathbb{C}^{d_A}}}{d_A} \otimes \frac{\mathbb{1}_{\mathbb{C}^{d_E}}}{d_E} \otimes T(X) = \frac{1}{d_E^2 d_A^2} \sum_{i=1}^{d_E^2} \sum_{j=1}^{d_A^2} \mathrm{Ad}_{\left(U_i^A \otimes V_j^E \otimes \mathbb{1}_{d_B}\right)} \circ \mathrm{Ad}_W\left(X \otimes |\psi_{EB}\rangle\langle\psi_{EB}|\right),$$

for some unitaries $\{U_1^A, \ldots, U_{d_A^2}^A\} \subset \mathcal{U}\left(\mathbb{C}^{d_A}\right)$ and $\{V_1^E, \ldots, V_{d_E^2}^E\} \subset \mathcal{U}\left(\mathbb{C}^{d_E}\right)$, and some unitary $W \in \mathcal{U}\left(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_E} \otimes \mathbb{C}^{d_B}\right)$.

Excercise 4 (Entanglement manipulation)**.** Let $\mathcal{H}_{A_1}, \mathcal{H}_{A_2}, \mathcal{H}_{B_1}, \mathcal{H}_{B_2}$ denote complex Euclidean spaces. A quantum channel

$$T : B(\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1}) \to B(\mathcal{H}_{A_2} \otimes H_{\mathcal{B}_2})$$

is called a *separable*[1] *quantum channel* if it can be written as

$$T = \sum_{n=1}^{N} S_A^{(n)} \otimes R_B^{(n)},$$

for completely positive maps $S_A^{(n)} : B(\mathcal{H}_{A_1}) \to B(\mathcal{H}_{A_2})$ and $R_B^{(n)} : B(\mathcal{H}_{B_1}) \to B(\mathcal{H}_{B_2})$. Let us denote the set of separable channels by

$$\mathrm{SepC}\left((\mathcal{H}_{A_1}, \mathcal{H}_{B_1}) \to (\mathcal{H}_{A_2}, \mathcal{H}_{B_2})\right).$$

(1) Show that $T(\rho_{AB}) \in \mathrm{Sep}(\mathcal{H}_{A_2}, \mathcal{H}_{B_2})$ for any $\rho_{AB} \in \mathrm{Sep}(\mathcal{H}_{A_1}, \mathcal{H}_{B_1})$ and any $\mathrm{SepC}((\mathcal{H}_{A_1}, \mathcal{H}_{B_1}) \to (\mathcal{H}_{A_2}, \mathcal{H}_{B_2}))$. The separable states are preserved under separable operations!

(2) Using the teleportation protocol, construct a separable quantum channel

$$T \in \mathrm{SepC}\left((\mathbb{C}^2 \otimes \mathbb{C}^2, \mathbb{C}^2) \to (\mathbb{C}^2 \otimes \mathbb{C}^2, \mathbb{C}^2)\right),$$

such that

$$T(\rho \otimes \omega_{AB}) = \frac{1}{4}(\mathbb{1}_{\mathbb{C}^2} \otimes \mathbb{1}_{\mathbb{C}^2}) \otimes \rho \in D(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2),$$

for any quantum state $\rho \in D(\mathbb{C}^2)$ and where $\omega_{AB} \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$ denotes the (normalized) maximally entangled state. How does this result generalize to dimensions $d > 2$.

(3) Show that any quantum state $\rho \in D(\mathbb{C}^d \otimes \mathbb{C}^d)$ can be written as

$$\rho = T(\omega_d),$$

for some separable quantum channel $T \in \mathrm{SepC}\left((\mathbb{C}^d, \mathbb{C}^d) \to (\mathbb{C}^d, \mathbb{C}^d)\right)$, and where $\omega_d \in D(\mathbb{C}^d \otimes \mathbb{C}^d)$ denotes the normalized maximally entangled state. This is why ω_d is called *maximally* entangled.

[1] Do not confuse this with separable quantum states!