MAT4430, Exercises-sheet 1, due 29.01.2026

Exercise (This exercise verifies the claims of Theorem 1.3 in the QIT notes (Lecture 1.) Let Σ_1, Σ_2 be non-empty alphabets (finite or countable), and suppose that (X_1, X_2) is a pair of random variables with values in $\Sigma_1 \times \Sigma_2$ distributed according to some $p_{1,2} \in \mathcal{P}(\Sigma_1 \times \Sigma_2)$.

Let $p_1 \in \mathcal{P}(\Sigma_1)$ and $p_2 \in \mathcal{P}(\Sigma_2)$ be the marginal distributions, as defined in Lecture 1. Recall that X_1, X_2 are random variables with probability distributions given by p_1, p_2, respectively. We have defined

$$P(X_1 = x_1, X_2 = x_2) := p_{1,2}(x_1, x_2)$$

for $(x_1, x_2) \in \Sigma_1 \times \Sigma_2$, where we interpret the left-hand side (LHS) as the probability that (X_1, X_2) takes the value (x_1, x_2). Similarly, for $S \subset \Sigma_1 \times \Sigma_2$, we let the probability that (X_1, X_2) takes a value in S be determined by

$$P((X_1, X_2) \in S) := \sum_{(x_1,x_2)\in S} p_{1,2}(x_1, x_2).$$

For $(x_1, x_2) \in \Sigma_1 \times \Sigma_2$ with $p_2(x_2) > 0$ we define the *conditional probability* that X_1 takes the value x_1 given the knowledge that X_2 takes the value x_2, denoted $P(X_1 = x_1|X_2 = x_2)$, by

$$P(X_1 = x_1|X_2 = x_2) := \frac{p_{1,2}(x_1, x_2)}{p_2(x_2)}.$$

For $S_j \subset \Sigma_j$, where $j = 1, 2$, we let

$$P(X_1 \in S_1|X_2 \in S_2) := \frac{\sum_{x_1\in S_1, x_2\in S_2} p_{1,2}(x_1, x_2)}{\sum_{x_2\in S_2} p_2(x_2)} \tag{1}$$

whenever the denominator, that is $P(X_2 \in S_2)$, is nonzero.

(i) Verify that the LHS of (1) defines a map $p_1(\cdot|X_2 \in S_2)$ in $\mathcal{P}(\Sigma_1)$. This is known as the *probability distribution of* X_1 *conditioned to* $X_2 \in S_2$, for every $S_2 \subset \Sigma_2$.

(ii) (Product rule). Show that

$$P(X_1 \in S_1, X_2 \in S_2) = P(X_2 \in S_2)P(X_1 \in S_1|X_2 \in S_2).$$

(iii) (The law of total probability.) Verify that

$$P(X_1 \in S_1) = \sum_{x_2\in\Sigma_2} P(X_1 \in S_1|X_2 = x_2)P(X_2 = x_2).$$

(iv) (Bayes' theorem.) Assume that $P(X_1 \in S_1) > 0$. Show that

$$P(X_2 \in S_2|X_1 \in S_1) = \frac{P(X_1 \in S_1|X_2 \in S_2)P(X_2 \in S_2)}{P(X_1 \in S_1)}.$$

(Note here that the denominator can be expressed according to (iii).)