MAT4430, Exercises-sheet 2, to be discussed 05.02.2026

Exercise 1. Verify the claim that $H(p) > m$ for every $m \geq 1$ from Example 1 in Lecture 4 .

Exercise 2. (This exercise gives a version of Jensen's inequality): Suppose that f is a convex function on $(0, \infty)$ and $d \in \mathbb{N}$. Show that for $t_1, t_2 \dots t_d \in (0, \infty)$ and $\lambda_1, \lambda_2, \dots, \lambda_d \in [0, 1]$ satisfying $\lambda_1 + \dots + \lambda_d = 1$, the following holds

$$f(\lambda_1 t_1 + \dots \lambda_d t_d) \leq \lambda_1 f(t_1) + \dots \lambda_d f(t_d).$$

Exercise 3. (i) Prove Lemma 3.6 in [1]. (Hint: Use exercise 2.)

(ii) Use part (i) to give another proof of the joint convexity of the relative entropy, see Theorem 3.7 in [1].

(iii) Show Gibb's inequality: given a finite non-empty alphabet Σ and $p, q \in \mathcal{P}(\Sigma)$, show that

$$D(p\|q) \geq 0,$$

with equality if and only if $p = q$.

(iv) Let $p \in \mathcal{P}(\Sigma)$ and let $u \in \mathcal{P}(\Sigma)$ be the uniform distribution given by

$$u(x) = \frac{1}{|\Sigma|} \text{ for each } x \in \Sigma.$$

Show that $H(p) = \log(|\Sigma|) - D(p\|q)$. Conclude that $H(p) \leq \log(|\Sigma|)$, with equality if and only if $p = u$.

References

[1] A. Muller-Hermes, Lecture notes on quantum information theory, 2025.