MAT4430, Exercises-sheet 4, to be discussed 24.02.2026

Exercise 1.

Let $\mathcal{H}$ be a complex Euclidean space (assumed of dimension $d \geq 1$). Suppose that $\{P_j\}_{j=1}^N$ are projections in $B(\mathcal{H})$ that add to the identity operator, so

$$\sum_{j=1}^{N} P_j = I_{\mathcal{H}}.$$

Show that the projections are mutually orthogonal, in the sense that $P_i P_j = 0$ whenever $i \neq j$ for $i, j = 1, \ldots, N$.

What is the maximal number of projections that can add up to the identity in $B(\mathcal{H})$?

Exercise 2. (Positivity).

(i) Let $\alpha \in \mathbb{C}$ and let $L \in B(\mathbb{C}^2)$ be given by the matrix

$$A = \begin{pmatrix} 1 & \bar{\alpha} \\ \alpha & 1 \end{pmatrix}.$$

Show that L is a positive (semidefinite) operator if and only if $|\alpha| \leq 1$.

(ii) Let $\mathcal{H}$ be a complex Euclidean space. Consider the direct sum Hilbert space $\mathcal{H} \oplus \mathcal{H}$ with inner product

$$\langle (a_1, b_1) | (a_2, b_2) \rangle = \langle a_1 | a_2 \rangle + \langle b_1 | b_2 \rangle$$

for $a_1, a_2, b_1, b_2 \in \mathcal{H}$. Given $X \in B(\mathcal{H})$, define an operator $L \in B(\mathcal{H} \oplus \mathcal{H})$ by the matrix

$$B = \begin{pmatrix} I_{\mathcal{H}} & X^\dagger \\ X & I_{\mathcal{H}} \end{pmatrix}.$$

Prove that L is positive (semidefinite) if and only if $\|X\| \leq 1$.

Exercise 3. (i) Let $\mathcal{H}$ be a complex Euclidean space, which we assume of dimension $d \geq 1$. Show that the map $\mathrm{Tr} : B(\mathcal{H}) \to \mathbb{C}$ defined by

$$\mathrm{Tr}(L) := \sum_{i=1}^{d} \langle x_i | L x_i \rangle, \tag{1}$$

where $L \in B(\mathcal{H})$ and $\{|x_1\rangle, \ldots, |x_d\rangle\}$ is an arbitrary orthonormal basis for $\mathcal{H}$ is well-defined, meaning that the number given by the right-hand side of the formula (1) does not depend on the choice of orthonormal basis. Show moreover that Tr is a linear map. (Hint: Using the definition of $\mathrm{Tr}(L)$ from (1), show that $\mathrm{Tr}(L) = \mathrm{Tr}(ULU^\dagger)$ for every unitary $U \in B(\mathcal{H})$.) We call Tr the *trace functional* on $B(\mathcal{H})$.

(ii) Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be complex Euclidean spaces of dimensions d_A and d_B, respectively. Show that $B(\mathcal{H}_A, \mathcal{H}_B)$ is a complex Hilbert space with the inner product defined by

$$\langle L_1, L_2 \rangle_{HS} := \mathrm{Tr}(L_1^\dagger L_2) \tag{2}$$

for $L_1, L_2 \in B(\mathcal{H}_A, \mathcal{H}_B)$, where Tr is the trace functional on $\mathcal{H}_A$.

(iii) With the notation from (ii), show that the set

$$\{|i\rangle\langle j| \mid 1 \leq i \leq d_A, 1 \leq j \leq d_B\}$$

forms an orthonormal basis for $B(\mathbb{C}^{d_A}, \mathbb{C}^{d_B})$ with the Hilbert-Schmidt inner product. Check that the matrix form of $|i\rangle\langle j|$ has entry 1 in the i'th row and j'th entry, with 0 in all other entries.

Exercise 4. Let $\{|1\rangle, |2\rangle, \ldots, |d\rangle\}$ denote the computational basis for $\mathbb{C}^d$, and likewise let $\{|1\rangle, |2\rangle, \ldots, |d'\rangle\}$ denote the computational basis for $\mathbb{C}^{d'}$, for $d, d' \geq 2$. Let

$$|ij\rangle := |i\rangle \otimes |j\rangle$$

for $1 \leq i \leq d, 1 \leq j \leq d'$. Consider the collection $\{|ij\rangle\}_{i,j}$ listed with the lexicographic order, meaning that ij appears before $i'j'$ provided that either $i < i'$ or $i = i'$ and $j < j'$, for all $i, j \in \{1, \ldots, d\}$.

(i) Show that $\{|ij\rangle\}_{i,j}$ is an orthonormal basis for $\mathbb{C}^{dd'}$.

(ii) Assuming $d = d' = 2$, write explicitly the basis for $\mathbb{C}^4$ obtained in this manner, you may denote it by $\{|\mathbf{1}\rangle, |\mathbf{2}\rangle, |\mathbf{3}\rangle, |\mathbf{4}\rangle\}$.

Define the operator $F : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ by

$$F(|a\rangle \otimes |b\rangle) := |b\rangle \otimes |a\rangle, \tag{3}$$

for $|a\rangle, |b\rangle \in \mathbb{C}^2$. Determine the matrix representation of F in the computational basis $\{|\mathbf{1}\rangle, |\mathbf{2}\rangle, |\mathbf{3}\rangle, |\mathbf{4}\rangle\}$.

(iii) Extend part (ii) to the operator $F : \mathbb{C}^d \otimes \mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^d$ given by the similar formula to (3).

Exercise 5. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be complex Euclidean spaces of dimensions d_A and d_B, respectively. Show that

$$B(\mathcal{H}_A \otimes \mathcal{H}_B) \cong B(\mathcal{H}_A) \otimes B(\mathcal{H}_B).$$