MAT4430, Exercises-sheet 5, to be discussed 05.03.2026

Exercise 1. Let $\mathcal{H} = \mathbb{C}^2$. Verify the claims in [1, Lecture 7] about the map $\mu := \mu_{\text{sic}}$ defined on $\Sigma = \{1, 2, 3, 4\}$. Thus:

(i) Show that μ defines a POVM that is not a PVM.

(ii) Show that $\{\mu(1), \mu(2), \mu(3), \mu(4)\}$ is a linear basis in $B(\mathbb{C}^2)$.

(iii) Verify that if $\rho, \sigma \in D(\mathbb{C}^2)$ are such that $\text{Tr}(\mu(k)\rho) = \text{Tr}(\mu(k)\sigma)$ for all $k = 1, \ldots, 4$, then $\rho = \sigma$.

Exercise 2. Let $\mathcal{H}$ be a complex Euclidean space of dimension $\dim(\mathcal{H}) \geq 2$. Show that

$$\frac{1}{\sqrt{\dim(\mathcal{H})}} |\Omega_{\mathcal{H}}\rangle$$

is a unit vector in $\mathcal{H} \otimes \mathcal{H}$, where $|\Omega_{\mathcal{H}}\rangle := \sum_{i=1}^{\dim(\mathcal{H})} |i\rangle \otimes |i\rangle$.

Exercise 3. Let $\mathcal{H}_A, \mathcal{H}_B$ be two complex Euclidean spaces, and let $X \in B(\mathcal{H}_A), Y \in B(\mathcal{H}_B)$.

(i) Show that $\text{Tr}(X \otimes Y) = \text{Tr}(X)\,\text{Tr}(Y)$. (Hint: use the Kronecker product form of $X \otimes Y$ with respect to the computational basis.)

(ii) Show that if X and Y are positive semidefinite, then so it $X \otimes Y$.

(iii) Show that $D(\mathcal{H}_A) \otimes D(\mathcal{H}_B) \subset D(\mathcal{H}_A \otimes \mathcal{H}_B)$. An element $\rho \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ is called a *product state* if it is of the form $\rho = \rho_A \otimes \rho_B$ for some $\rho_A \in D(\mathcal{H}_A)$ and $\rho_B \in D(\mathcal{H}_B)$.

(iv) Let $D(\mathcal{H}_A) = D(\mathcal{H}_B) = \mathbb{C}^2$, let $\{|0\rangle, |1\rangle\}$ be the computational basis, and define rank-one operators by $e_{i,j} := |i\rangle\langle j|$, for $i, j = 0, 1$. Show that

$$\rho = \frac{1}{4} e_{0,0} \otimes e_{0,0} + \frac{1}{4} e_{0,0} \otimes e_{1,1} + \frac{1}{4} e_{1,1} \otimes e_{0,0} + \frac{1}{4} e_{1,1} \otimes e_{1,1}$$

is a product state in $D(\mathbb{C}^2 \otimes \mathbb{C}^2)$. Express it in matrix form.

(v) Show that

$$\eta := \frac{1}{2} e_{0,0} \otimes e_{0,0} + \frac{1}{2} e_{1,1} \otimes e_{1,1}$$

is not a product state. Express it in matrix form.

(vi) Show that

$$\upsilon := \frac{1}{2} e_{0,0} \otimes e_{0,0} + \frac{1}{2} e_{0,1} \otimes e_{0,1} + \frac{1}{2} e_{1,0} \otimes e_{1,0} + \frac{1}{2} e_{1,1} \otimes e_{1,1}$$

is not a product state. Express it in matrix form.

References

[1] A. Muller-Hermes, Lecture notes on quantum information theory, 2025.