MAT4430, Exercises-sheet 6, to be discussed 12.03.2026

Exercise 1. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be complex Euclidean spaces. Suppose that $\rho \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ is such that

$$\rho = \sum_{\nu \in \Pi} X_\nu \otimes Y_\nu$$

for some finite alphabet Π and positive semidefinite operators $X_\nu \in B(\mathcal{H}_A)$, $Y \in B(\mathcal{H}_B)$.

(i) Show that ρ is a separable state. Thus, show that there exists a finite alphabet Σ, a probability distribution $p \in \mathcal{P}(\Sigma)$ and quantum states $\sigma_A^i \in D(\mathcal{H}_A)$, $\tau_B^i \in D(\mathcal{H}_A)$, for all $i = 1, \ldots, K$, where $K = |\Sigma|$, such that

$$\rho = \sum_{i=1}^{K} p_i \sigma_A^i \otimes \tau_B^i.$$

(Hint: Apply the spectral theorem to X_ν, Y_ν for each $\nu \in \Pi$.)

(ii) Show that $\rho \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ is separable if and only if there exist collections of unit vectors $|x_i\rangle \in \mathcal{H}_A$, $|y_i\rangle \in \mathcal{H}_B$ for all $i = 1, \ldots K$ such that

$$\rho = \sum_{i=1}^{K} p_i |x_i\rangle\langle x_i| \otimes |y_i\rangle\langle y_i|.$$

Exercise 2. Let $\mathcal{H}_A$ be a complex Euclidean space, assumed of the form $\mathbb{C}^{d_A}$. Define the operator $F : \mathcal{H}_A \otimes \mathcal{H}_A \to \mathcal{H}_A \otimes \mathcal{H}_A$ by

$$F(|a\rangle \otimes |b\rangle) := |b\rangle \otimes |a\rangle$$

for all $|a\rangle, |b\rangle \in \mathcal{H}_A$. This is the flip, or swap, operator.

(i) Show that F is a self-adjoint unitary. Show that it can be written as

$$F = \sum_{i,j=1}^{d_A} e_{i,j} \otimes e_{j,i},$$

where $e_{i,j} = |i\rangle\langle j|$ are the matrix units, for $i, j = 1, \ldots, d_A$.

(ii) Show that F has eigenvalues 1 and -1 and describe its eigenspaces. (Hint: What is the form of vectors spanning the two eigenspaces?)