MAT4430, Exercises-sheet 7, to be discussed 26.03.2026

Exercise 1. (Isometries between Hilbert spaces.) Let $\mathcal{H}, \mathcal{K}$ be complex Euclidean spaces. Show that if $V : \mathcal{H} \to \mathcal{K}$ is a linear map such that

$$\langle Vx|Vy\rangle = \langle x|y\rangle$$

for all $x, y \in \mathcal{H}$, then $\|Vx\| = \|x\|$ for all $x \in \mathcal{H}$. Show that the converse holds. Show that $V \in B(\mathcal{H}, \mathcal{K})$. (The map V is called an isometry).

(i) Show that if V is an isometry, then $V^\dagger V = 1_{\mathcal{H}}$, the identity operator in $B(\mathcal{H})$ and $VV^\dagger$ is a projection in $B(\mathcal{K})$.

(iii) Suppose that $\dim(\mathcal{H}) \leq \dim(\mathcal{K})$.

Let $\{|v_i\rangle\}_{i=1}^k \subset \mathcal{H}$ and $\{|w_i\rangle\}_{i=1}^k \subset \mathcal{K}$ be orthonormal sets of vectors. Show that there is an isometry $V : \mathcal{H} \to \mathcal{K}$ such that $V|v_i\rangle = |w_i\rangle$ for all $i = 1, \ldots, k$.

Exercise 2. Let $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_E$ be complex Euclidean spaces and $V \in B(\mathcal{H}_A, \mathcal{H}_A \otimes \mathcal{H}_E)$ be an isometry. Define a map $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ by

$$T(X) = \mathrm{Tr}_E[VXV^\dagger],$$

for each $X \in B(\mathcal{H}_A)$. Prove that T is a quantum channel.

Exercise 3. Let $\mathcal{H}_A, \mathcal{H}_E, \mathcal{H}_C$ be complex Euclidean spaces. Suppose that $|\psi\rangle\langle\psi| \in D(\mathcal{H}_A \otimes \mathcal{H}_E)$ is a purification of a quantum state $\rho \in D(\mathcal{H}_A)$. Suppose moreover that $\sigma \in D(\mathcal{H}_A \otimes \mathcal{H}_C)$ satisfies $\mathrm{Tr}_C[\sigma] = \rho$. Show that there exists a quantum channel $S : B(\mathcal{H}_E) \to B(\mathcal{H}_C)$ satisfying

$$(\mathrm{Id}_A \otimes S)(|\psi\rangle\langle\psi|) = \sigma.$$

Hint: Reduce the problem to the case where σ is a pure state.

Exercise 4. Compute the correlations from Example 10.8 in [1]. Show first that

$$P(a, b|A_i, B_j) = \frac{1}{2}\langle \beta_j(b)\alpha_i(a)\rangle_{HS},$$

for each $a, b \in \{\pm\}, i, j = 1, 2$. (Note that we used notation μ_i, ν_j in the lectures.) Hint: Use the necklace identity,

References

[1] A. Muller-Hermes, Lecture notes on quantum information theory, 2025.