MAT4430, Exercises-sheet 8, to be discussed 09.04.2026

Exercise 1. (The Weyl-Heisenberg system, a generalization of the Pauli matrices to higher dimensions.) Let $\mathbb{C}^d$ for $d \geq 2$ have the computational basis $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$. We endow $B(\mathbb{C}^d)$ with the Hilbert-Schmidt inner-product $\langle X, Y\rangle = \mathrm{Tr}[X^\dagger Y]$ for $X, Y \in B(\mathbb{C}^d)$.

(1) Define an operator $S : \mathbb{C}^d \to \mathbb{C}^d$ by

$$S|k\rangle = |k+1\rangle$$

for every $k = 0, \ldots, d-1$, where addition is mod d. Show that $\{S^k \mid k = 0, \ldots, d-1\}$ form a family of pairwise orthogonal elements with respect to the Hilbert-Schmidt inner product.

(2) Define the complex number $\omega_d = \exp\left(\frac{2\pi i}{d}\right)$. This satisfies $\omega_d^d = 1$ and is known as a dth root of unity. Form the diagonal matrix D in $B(\mathbb{C}^d)$ with diagonal list of entries given as

$$1, \omega_d, \omega_d^2, \ldots, \omega_d^{d-1}.$$

Show that the powers D^l, with $l = 0, \ldots, d-1$ are orthogonal with respect to the Hilbert-Schmidt inner product.

(3) Define unitary operators $U_{kl} = S^k D^l$ for all $k, l = 0, \ldots, d-1$, and show that they form an orthogonal basis for $B(\mathbb{C}^d)$ with the Hilbert-Schmidt inner product.

(4) Check that these operators extend the known basis in the case $d = 2$.