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[Analysis Now, Exercise 3.1.9]

Let $\{e_n | n \in \mathbb{N}\}$ be an orthonormal basis for the Hilbert space $\mathcal{H}$, and set $\mathcal{T} = \{\sqrt{n}e_n | n \in \mathbb{N}\}$. Show that $0 \in \overline{\mathcal{T}}^w$ but that no sequence from $\mathcal{T}$ converges to 0 weakly. Conclude that the weak topology on $\mathcal{H}$ does not satisfy the first countability axiom (hence it is not metrizable).

A suggested solution (others are possible)

Assume, for a contradiction, that $0 \notin \overline{\mathcal{T}}^w$. Then there is some basic open set U around 0 s.t. $U \cap \mathcal{T} = \emptyset$. We may assume U of the form $\bigcap_{j=1}^N \{x \in \mathcal{H} \mid |(x|y_j)| < \varepsilon_0\}$ for some $\varepsilon_0 > 0$

and some $y_1, \dots, y_N \in \mathcal{H}$. Thus we have:

there is $j=1, \dots, N$ s.t. $|(\sqrt{n}e_n | y_j)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$. Then

$$\infty > \|y_j\|^2 \geq \sum_{n=1}^{\infty} |(e_n | y_j)|^2 \geq \sum_{n=1}^{\infty} \frac{\varepsilon_0^2}{n} \rightarrow \infty, \text{ which is absurd. Therefore,}$$

$$0 \in \overline{\mathcal{T}}^w.$$

Assume again, for a contradiction, that there is $\{x_m\}_{m \in \mathbb{N}} \subseteq \mathcal{T}$

so that $\lim_m x_m = 0$ in the weak-topology on $\mathcal{H}$. In other

words, we assume that $\lim_{m \rightarrow \infty} (x_m | y) = 0$ for all $y \in \mathcal{H}$.

In particular, for every $y \in \mathcal{H}$, $\{(x_m | y)\}_m$ is a bounded sequence. Form linear operators $T_m: \mathcal{H} \rightarrow \mathbb{C}$, $T_m(y) = (y | x_m)$, with

$$\|T_m(y)\| = |(y | x_m)| < \infty. \text{ By the principle of uniform boundedness}$$

the set $\{\|T_m\|\}_{m \in \mathbb{N}}$ is bounded. However, $\|T_m\| = \|x_m\| \in \{\sqrt{n} | n \in \mathbb{N}\}$.

Exercise 3.1.9 continued

Assume for a contradiction that the neighborhood basis at 0 for the weak topology contains a countable basis $\{U_n\}_{n \geq 1}$.

Pick a net $\{x_\lambda\}_{\lambda \in \Lambda}$ in T that converges weakly to 0.

For each $n \geq 1$, $\{x_\lambda\}_\lambda$ is eventually in U_n , thus there is $\lambda(n) \in \Lambda$

s.t. $x_\lambda \in U_n$ for $\lambda(n) \leq \lambda$. Form $\{\tilde{x}_n = x_{\lambda(n)}\}_{n \in \mathbb{N}}$ a sequence in T , which by construction will be weakly convergent to 0, contradicting the earlier part of the exercise. Thus no such countable basis can exist.

Exercise

Suppose that X is an infinite dimensional normed space.

(a) Show that if V is a weakly open set containing 0 , then V contains an infinite dimensional subspace of X .

(b) Similarly, if U is a weak^{*} open set containing 0 , then U contains an infinite dimensional subspace of X^* .

Suggested solution for (a)

We may find $V_0 = \bigcap_{j=1}^n \{x \in X \mid |\varphi_j(x)| < \varepsilon\}$ for $\varphi_1, \dots, \varphi_n \in X^*$

with $0 \in V_0 \subseteq V$. Form $T: X \rightarrow \mathbb{C}^n$, $Tx = (\varphi_1(x), \dots, \varphi_n(x))$.

Then T is linear. Clearly, $\ker T \subseteq V_0$. Since $T(X)$ is a (finite dimensional) subspace of $\mathbb{C}^n$ and since $\ker T$ is a subspace of X , we conclude that $\ker T$ is an infinite dimensional subspace of V_0 .

(b) This is similar.