

Bounded operators on Hilbert spaces

Let $T: \mathcal{H} \rightarrow \mathcal{H}$ be a linear map (linear operator) on the Hilbert space $\mathcal{H}$ with values in $\mathcal{H}$. Then T is bounded iff $\|T\| < \infty$, where

$$\|T\| = \sup\{\|Tx\| \mid x \in \mathcal{H}, \|x\| \leq 1\}. \text{ We assume } \mathbb{F} = \mathbb{C}.$$

Equivalently, T is bounded iff T is continuous iff T is continuous at 0. Let $B(\mathcal{H})$ denote the space of bounded linear operators on $\mathcal{H}$.

Fact: To a bounded $T \in B(\mathcal{H})$ corresponds a map

$$B_T: \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}, \quad B_T(x, y) = (x | Ty),$$

where $(\cdot | \cdot)$ denotes the inner product on $\mathcal{H}$, which is linear in the first coordinate and conjugate linear in the second variable. Thus,

$$B_T(\alpha x + \beta w, y) = \alpha B_T(x, y) + \beta B_T(w, y)$$

$$\text{and } B_T(x, \alpha y + \beta w) = \bar{\alpha} B_T(x, y) + \bar{\beta} B_T(x, w), \quad \forall x, y, w \in \mathcal{H}, \alpha, \beta \in \mathbb{C}.$$

$$\text{Further, } \|B_T\| \stackrel{\text{def}}{=} \sup\{|B_T(x, y)| \mid \|x\| \leq 1, \|y\| \leq 1\}$$

$$= \sup\{|(x | Ty)| \mid \|x\| \leq 1, \|y\| \leq 1\}$$

$$\leq \|T\| \text{ by the Cauchy-Schwartz inequality for } (\cdot | \cdot).$$

$$\text{Since } \|Tx\|^2 = (Tx | Tx) = B_T(Tx, x) = |B_T(Tx, x)| \leq \|B_T\| \|Tx\| \|x\| \leq \|B_T\| \|T\| \|x\|^2, \text{ we get } \|T\|^2 \leq \|B_T\| \|T\|, \text{ so } \|T\| \leq \|B_T\|.$$

In all, $\|T\| = \|B_T\|$, so B_T is bounded with same norm as T .

Conversely, if $B: \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ is linear in 1st variable, conjugate linear in 2nd variable, and bounded, then $B(\cdot, y): \mathcal{H} \rightarrow \mathbb{C}$ is bounded for each $y \in \mathcal{H}$, so by the Riesz' lemma $\exists$ unique vector in $\mathcal{H}$, denoted Ty , s.t. $B(x, y) = (x | Ty)$. Then T is linear as a map $T: \mathcal{H} \rightarrow \mathcal{H}$ (Exercise) and bounded, with

$\|T\| \leq \|B\|$. The correspondence $T \rightarrow B_T$ is therefore bijective

and preserves the norms, since $\|T\| = \|B_T\|$. Recall that each $T \in B(\mathcal{H})$ admits a unique adjoint, denoted T^* , such that $(Tx | y) = (x | T^*y)$, $\forall x, y \in \mathcal{H}$, $T^* \in B(\mathcal{H})$.

②

Positive bounded operators

An element $T \in B(H)$ is positive, in notation $T \geq 0$, if $T = T^*$ (so T is self-adjoint) and (1) $(Tx|x) \geq 0 \quad \forall x \in H$.

In fact, since $F = \mathbb{C}$, it holds true that $(Tx|x) \in \mathbb{R} \quad \forall x \in H \Leftrightarrow T = T^*$, so (1) suffices.

Partial order on $B(H)_{sa} = \{T \in B(H) \mid T \text{ self-adjoint}\}$ by
 $S \leq T \stackrel{\text{def.}}{\iff} T - S \geq 0$. Let $I \in B(H)_{sa}$ be the identity operator.

Proposition ([P, Prop. 3.2.9]) The following hold true for $S, T \in B(H)_{sa}$.

- (a) $S \leq T$ in $B(H)_{sa} \Rightarrow A^*SA \leq A^*TA, \quad \forall A \in B(H)$.
- (b) $0 \leq S \leq T \Rightarrow \|S\| \leq \|T\|$.
- (c) $0 \leq S \leq I \iff \|S\| \leq 1$ and $0 \leq S$.
- (d) If $T \in B(H)_{sa}$, then $-I \leq T \leq I$ if and only if $\|T\| \leq 1$.

Proof:

(a) Let $S \leq T$, then $\forall x \in H, (T-S)y|y) \geq 0$. Given $A \in B(H), x \in H$, we get $(T-S)Ax|Ax) \geq 0$.

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$$(A^*(T-S)Ax|x) = (A^*TAx - A^*SAx|x) = ((A^*TA - A^*SA)x|x).$$

So $A^*SA \leq A^*TA$.

(b) Let $0 \leq S \leq T$. Consider $B_{S^*}: H \times H \rightarrow \mathbb{C}, B_{S^*}(x, y) = (x|S^*y)$. Then $B_{S^*}(x, x) = (x|S^*x) = (Sx|x) \geq 0, \quad \forall x \in H$. Further, B_{S^*} is linear in 1st variable and conjugate linear in 2nd variable. Thus, B_{S^*} satisfies the Cauchy-Schwartz inequality:

$$|B_{S^*}(x, y)| \leq B_{S^*}(x, x)^{1/2} B_{S^*}(y, y)^{1/2} \quad \forall x, y \in H.$$

This means $|(Sx|y)| \leq (Sx|x)^{1/2} (Sy|y)^{1/2}, \quad \forall x, y \in H$.

From $T - S \geq 0$ we obtain $|(Sx|y)| \leq (Tx|x)^{1/2} (Ty|y)^{1/2}$.

By the Cauchy-Schwarz inequality for the inner-product, ③
 $0 \leq (Tx|x) \leq \|Tx\| \cdot \|x\| \leq \|T\| \cdot \|x\|^2, \forall x \in \mathcal{H}$ (Similar for $(Ty|y)$).

$\Rightarrow |(Sx|y)|^2 \leq \|T\|^2 \cdot \|x\|^2 \cdot \|y\|^2$ so taking sup as $\|x\| \leq 1, \|y\| \leq 1$ implies $\|S\| \leq \|T\|$.

(c) The forward implication is from (b) applied with $T=I$.

Assume $S \geq 0$ and $\|S\| \leq 1$. For $x \in \mathcal{H}$ with $\|x\| \leq 1$

$$((I-S)x|x) = (x|x) - (Sx|x) = 1 - (Sx|x) \geq 0 \text{ since } \|B_{S^*}\| = \|S^*\| = \|S\|.$$

If $x \in \mathcal{H} \setminus \{0\}$, replace it with $\frac{x}{\|x\|}$ in the previous line. If $x=0$, clearly $((I-S)x|x) = 0 \geq 0$. In all, $0 \leq S \leq I$.

(d) Suppose $T=T^*$ and $-I \leq T \leq I$. Let $x, y \in \mathcal{H}, \|x\| = \|y\| = 1$.

Since $T+I \geq 0$, we have $((T+I)(x-y)|x-y) \geq 0$

Since $I-T \geq 0$, we have $((I-T)(x+y)|x+y) \geq 0$.

$$\text{So, } (Tx|x) - (Tx|y) + (x|x) - (x|y) - (Ty|x) + (Ty|y) - (y|x) + (y|y) \geq 0$$

$$(x|x) + (x|y) + (y|x) + (y|y) - (Tx|x) - (Tx|y) - (Ty|x) - (Ty|y) \geq 0$$

Add them: $2\|x\|^2 + 2\|y\|^2 - 2(Tx|y) - 2(Ty|x) \geq 0$. Recall that $T=T^*$.

$$\text{So } (Tx|y) + \overline{(Tx|y)} \leq 2 \text{ or } 2\operatorname{Re}(Tx|y) \leq 2, \forall x, y \in \mathcal{H}.$$

$$\text{In all, } \|B_{T^*}\| = \|B_T\| = \|T\| \leq 1.$$

Now let $T=T^*$ s.t. $\|T\| \leq 1$. Then $|(Tx|x)| \leq \|x\|^2, \forall x \in \mathcal{H}$,

$$\text{so } -\|x\| \leq (Tx|x) \leq \|x\|, \forall x \in \mathcal{H}$$

$\Leftrightarrow -(x|x) \leq (Tx|x) \leq (x|x), \forall x \in \mathcal{H}$. This means $-I \leq T \leq I$.

Existence of square root of a positive bounded operator

(4)

First (elementary) proof:

Lemma (Real Analysis)

There is a sequence of polynomials $\{p_n\}_n$ with positive coefficients s.t. $\sum_n p_n$ converges uniformly on $[0, 1]$ to the function $f(t) = 1 - (1-t)^{1/2}$.

Proof Define $q_0 = 0$, $q_1(t) = \frac{1}{2}t$, $q_n(t) = \frac{1}{2}(t + q_{n-1}(t)^2)$, $n \geq 2$.

By induction, q_n has positive coefficients and $q_n(t) \leq 1$, $\forall t \in [0, 1]$.

From

$$2(q_{n+1}(t) - q_n(t)) = (t + q_n(t)^2) - (t + q_{n-1}(t)^2) = (q_n(t) - q_{n-1}(t))(q_n(t) + q_{n-1}(t)),$$

with $p_n(t) = q_n(t) - q_{n-1}(t)$ for $n \geq 1$, we see

$$2p_{n+1}(t) = p_n(t) \cdot (q_n(t) + q_{n-1}(t))$$

has positive coefficients.

In all, $p_n(t)$ has positive coefficients, hence $\{q_n(t)\}_n$ is monotone increasing.

With $q_n \in C[0, 1]$, $\forall n \geq 0$, we have that q_n converges pointwise to a function $g: [0, 1] \rightarrow \mathbb{R}$ s.t. in the limit as $n \rightarrow \infty$,

$$g(t) = \frac{1}{2}(t + g(t)^2).$$

Then $g(t) = 1 - \sqrt{1-t^2}$ (from $a^2 - 2a + t = 0$, $t \in [0, 1]$, $g(t) \in C[0, 1]$).

By Dini's lemma, q_n converges uniformly to g on $[0, 1]$, and

$$\sum_n p_n = \lim_{N \rightarrow \infty} \sum_1^N p_n = \lim_{N \rightarrow \infty} \sum_1^N (q_n - q_{n-1}) = \lim_{N \rightarrow \infty} q_n = g.$$

(5)

Proposition

Given $T \in B(\mathcal{H})$, $T \geq 0$, there is a unique positive operator in $B(\mathcal{H})$, denoted $T^{1/2}$, s.t. $(T^{1/2})^2 = T$. Further, if $BT = TB$ for some $B \in B(\mathcal{H})$, then $BT^{1/2} = T^{1/2}B$.

Proof:

Assume first $T \leq I$, and let $S = I - T$. Then $0 \leq S \leq I$. For each $n \geq 0$, let p_n be the polynomials given by the previous lemma. So, p_n has form $p_n = \sum_{k=0}^{m_n} a_{k,n} t^k$, $a_{k,n} \geq 0$, $m_n \in \mathbb{N}$.

Given $\varepsilon > 0$, the uniform convergence of $\sum_{n=0}^{\infty} p_n$ on $[0, 1]$ implies that there is $N \in \mathbb{N}$ s.t. $m \geq N \Rightarrow 0 \leq \sum_{n=N}^m p_n(t) < \varepsilon$ uniformly on $[0, 1]$.

Note that $p_n(S) = \sum_{k=0}^{m_n} a_{k,n} S^k$ is again an element of $B(\mathcal{H})$.

$$\text{Further, } \left\| \sum_{n=N}^m p_n(S) \right\| = \left\| \sum_{n=N}^m a_{k,n} S^k \right\| \leq \sum_{n=N}^m a_{k,n} \cdot \|S\|^k \leq \sum_{n=N}^m a_{k,n}$$

$$\|S^k\| \leq \|S\|^k$$

$$\|S\| \leq 1$$

by previous ^{proposition} lemma. In all,

(*) $\sum p_n(S)$ converges in $B(\mathcal{H})$.

Let $R = \lim \sum p_n(S)$ in $B(\mathcal{H})$. By previous ^{proposition} lemma, $0 \leq R \leq I$, since $\|R\| \leq 1$ because $\sum p_n$ converges uniformly to $g(t)$ on $[0, 1]$ and $g(t) \leq 1$ on $[0, 1]$. Moreover,

$$\begin{aligned} (I-R)^2 &= \left(I - \sum_n p_n(S) \right)^2 = \lim_{n \rightarrow \infty} \left(I - \sum_n p_n(S) \right)^2 = \lim_n (I - q_n(S))^2 \\ &= \lim_n (I - 2q_n(S) + q_n(S)^2) = \lim_n (I - 2q_n(S) + 2q_{n+1}(S) - S) \\ &= I - S + \lim_n 2(q_n(S) - q_{n+1}(S)) = I - S = T. \end{aligned}$$

We may let $T^{1/2} = I - R \geq 0$. This shows existence when $T \leq I$. For arbitrary T , replace it by $\|T\|^{-1} T$, assuming $T \neq 0$.