

MAT 4450 Advanced Functional Analysis

Lecture 12

23/02/2022

(1)

Existence and uniqueness of the square root of a positive operator-continued

Proposition If $T \in B(\mathcal{H})$, $T \geq 0$, then there is a unique positive operator in $B(\mathcal{H})$, denoted $T^{1/2}$, such that $(T^{1/2})^2 = T$. Further, if $BT = TB$ for $B \in B(\mathcal{H})$, then $BT^{1/2} = T^{1/2}B$.

Proof:

We have established existence of an operator $T^{1/2} \geq 0$ with $(T^{1/2})^2 = T$. (By approximating the function $f(t) = 1 - \sqrt{1-t}$ on $[0,1]$ uniformly as $\sum_{n=0}^{\infty} p_n$, with $p_n(t) \in C[0,1]$ polynomials with positive coefficients.)

Assume $BT = TB$. Then, $B(I-S) = (I-S)B$ so $BS = SB$. For $p_n(t)$ as in the proof of existence, it follows that $B(\sum_{k=0}^{m_n} \alpha_{n,k} S^k) = (\sum_{k=0}^{m_n} \alpha_{n,k} S^k)B$, and by continuity of product in $B(\mathcal{H})$ we get $B \sum_{n=0}^{\infty} p_n(S) = (\sum_{n=0}^{\infty} p_n(S))B$.

In all, $BT^{1/2} = T^{1/2}B$.

For uniqueness, suppose that $A \in B(\mathcal{H})$ satisfies $A \geq 0$ and $A^2 = T$.

Then $AT = AA^2 = A^3 = TA$, therefore $AT^{1/2} = T^{1/2}A$. Hence, we have $(T^{1/2} - A)(T^{1/2} + A) = 0$ in $B(\mathcal{H})$.

It follows that $(T^{1/2} - A)((T^{1/2} + A)x) = 0$ for all $x \in \mathcal{H}$. By continuity of product, we get $(T^{1/2} - A)y = 0$ for all $y \in \overline{(T^{1/2} + A)\mathcal{H}}^{||\cdot||}$. Denote $Y = \overline{(T^{1/2} + A)\mathcal{H}}^{||\cdot||}$.

Recall that for $M \subseteq \mathcal{H}$ subset, $M^\perp = \{z \in \mathcal{H} \mid (m|z) = 0, \forall m \in M\}$ is a closed subspace of $\mathcal{H}$, called the annihilator of M in $\mathcal{H}$.

We have $M^{\perp\perp} = \overline{M}^{||\cdot||}$ for every subspace M of $\mathcal{H}$. With $M = (T^{1/2} + A)\mathcal{H}$, we also have that $\ker(T^{1/2} + A) = M^\perp$ (direct verification, using that $(T^{1/2} + A)^* = T^{1/2} + A$). So $Y = \overline{M}^{||\cdot||} = M^{\perp\perp} = \ker(T^{1/2} + A)^\perp$.

Since $\mathcal{H} = Y \oplus Y^\perp$ (orthogonal decomposition), it remains to show:

$$(T^{1/2} - A)z = 0 \text{ for } z \in Y^\perp = \ker(T^{1/2} + A)^{\perp\perp} = \ker(T^{1/2} + A). \quad \textcircled{2}$$

$$(\text{Since } \ker(T^{1/2} + A) = \overline{\ker(T^{1/2} + A)}^{\|\cdot\|}).$$

It will then follow: $(T^{1/2} - A)x = 0$, $\forall x = y + z \in Y \oplus Y^\perp = \mathcal{H}$.
Let therefore $z \in \ker(T^{1/2} + A)$. We get, using $T^{1/2} \leq T^{1/2} + A$,

$$((T^{1/2})z | z) \leq ((T^{1/2} + A)z | z) = (0 | z) = 0. \text{ Then}$$

$$\begin{aligned} \|(T^{1/2})^{1/2}z\|^2 &= ((T^{1/2})^{1/2}z | (T^{1/2})^{1/2}z) \\ &= ((T^{1/2})^{1/2}(T^{1/2})^{1/2}z | z) \\ &= (T^{1/2}z | z) = 0, \text{ so } (T^{1/2})^{1/2}z = 0 \text{ for } z \in Y^\perp. \end{aligned}$$

Therefore $T^{1/2}z = (T^{1/2})^{1/2}(T^{1/2})^{1/2}z = 0$ for $z \in Y^\perp = \ker(T^{1/2} + A)$. But for such z , we already have $(T^{1/2} + A)z = 0$, so necessarily $Az = 0 = T^{1/2}z$. $\square$.

Example:

$\mathcal{H} = L^2[0,1]$, define $T: \mathcal{H} \rightarrow \mathcal{H}$, $(Tf)(x) = x^2 f(x)$ for $f \in L^2[0,1]$.
Then $(Tf|f) = \int_{[0,1]} Tf(x) \overline{f(x)} d\lambda(x) = \int_{[0,1]} x^2 f(x) \overline{f(x)} d\lambda(x) = \int_{[0,1]} x^2 |f(x)|^2 d\lambda(x) \geq 0$
for all $f \in L^2[0,1]$. Hence $T \geq 0$.

Check directly that $(T^{1/2}f)(x) = x f(x)$ is the square root.
(Enough to verify that $(T^{1/2})^2 = T$, because by applying the uniqueness we conclude this must be the square root.)

Remark: If $A, B \in \mathcal{B}(\mathcal{H})$, $A, B \geq 0$, $AB = BA$ (A and B commute), then $AB \geq 0$. Indeed,

$$AB = (A^{1/2})(A^{1/2}B) = A^{1/2}(BA^{1/2}) = (A^{1/2})^* B A^{1/2} \geq 0 \text{ by earlier proposition.}$$

In general, without assuming A, B commute, we cannot conclude that $AB \geq 0$ when $A, B \geq 0$.