

MAT4450 - Spring 2026 - Exercises - Set 10

We recall that our convention that by a unital Banach algebra we mean a complex unital Banach algebra whose unit has norm equal to 1.

Exercise 45 [Will be a part of the compulsory assignment]

Let $\mathcal{A}$ be a *commutative* unital Banach algebra. Define the *radical* of $\mathcal{A}$ by

$$\text{rad}(\mathcal{A}) = \bigcap_{\gamma \in \widehat{\mathcal{A}}} \ker(\gamma).$$

We will say that $\mathcal{A}$ is *semisimple* when $\text{rad}(\mathcal{A}) = \{0\}$.

- Let Ω be a topological space. Check that $C_b(\Omega)$ is semisimple.
- Show that $\mathcal{A}$ is semisimple if and only if it holds that $a = 0$ whenever $a \in \mathcal{A}$ and $r_{\mathcal{A}}(a) = 0$.
- Let $\mathcal{A}$ be a semisimple commutative unital Banach algebra and $\mathcal{B}$ be unital Banach algebra. Assume $\Phi : \mathcal{B} \rightarrow \mathcal{A}$ is an algebra-homomorphism mapping the unit of $\mathcal{B}$ to the unit of $\mathcal{A}$. Show that Φ is norm-continuous.

Hint: Use the closed graph theorem.

- Deduce that on a semisimple commutative unital Banach algebra all Banach algebra norms are equivalent.

Exercise 46

Consider the unital commutative Banach algebra $\mathcal{A} = \ell^1(\mathbb{Z}, \mathbb{C})$ (w.r.t. the $\|\cdot\|_1$ -norm). Set $\mathcal{B} := \{f \in \mathcal{A} : f(n) = 0 \text{ for all } n < 0\}$.

- Check that $\mathcal{B}$ is a norm-closed subalgebra of $\mathcal{A}$ which contains the unit of $\mathcal{A}$. (Hence $\mathcal{B}$ is also a unital commutative Banach algebra.)
- Let $\lambda \in \mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$. Check that the map $\gamma_\lambda : \mathcal{B} \rightarrow \mathbb{C}$ given by

$$\gamma_\lambda(f) = \sum_{n=0}^{\infty} f(n)\lambda^n$$

for all $f \in \mathcal{B}$ is well-defined and belongs to $\widehat{\mathcal{B}}$.

- Show that the map $\lambda \mapsto \gamma_\lambda$ gives a homeomorphism from $\mathbb{D}$ onto $\widehat{\mathcal{B}}$. Then, identifying $\widehat{\mathcal{B}}$ with $\mathbb{D}$, check that the Gelfand transform of $\mathcal{B}$ is the map $\Gamma : \mathcal{B} \rightarrow C(\mathbb{D})$ given by

$$[\Gamma(f)](\lambda) = \sum_{n=0}^{\infty} f(n)\lambda^n \quad \text{for } f \in \mathcal{B} \text{ and } \lambda \in \mathbb{D}.$$

Deduce that Γ is one-to-one and describe the range of Γ .

Exercise 47 [Will be a part of the compulsory assignment]

Let $N \in \mathbb{N}$ and let $\mathbb{Z}_N = \{0, 1, \dots, N-1\}$ be the cyclic group of order N (addition being defined modulo N). Consider the Banach space $\mathcal{A} = \ell^1(\mathbb{Z}_N, \mathbb{C})$ with the $\|\cdot\|_1$ -norm. It can easily be checked that $\mathcal{A}$ becomes a commutative unital Banach algebra w.r.t. the convolution product defined for every $f, g \in \mathcal{A}$ by

$$(f * g)(n) := \sum_{m \in \mathbb{Z}_N} f(m)g(n-m) \quad \text{for all } n \in \mathbb{Z}_N.$$

Explain that the character space $\widehat{\mathcal{A}}$ of $\mathcal{A}$ can be identified with $\mathbb{Z}_N$. Then describe the Gelfand transform $\Gamma : \mathcal{A} \rightarrow C(\widehat{\mathcal{A}})$ as best you can.