

MAT4450 - Spring 2026 - Exercises - Set 11

Recall that if Ω is a topological space, then $C_b(\Omega)$ consists of all complex bounded continuous functions on Ω . When equipped with its natural pointwise defined operations and the $\|\cdot\|_\infty$ -norm, we have seen that $C_b(\Omega)$ is a unital commutative Banach algebra. Letting involution being defined as conjugation of functions, it is routine to check that $C_b(\Omega)$ is a unital commutative C^* -algebra. Of course, if Ω is a compact Hausdorff space, then $C_b(\Omega) = C(\Omega)$, whose character space may be identified with Ω .

We also recall that by a $*$ -homomorphism $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ between two $*$ -algebras, we mean an (algebra) homomorphism satisfying that $\Phi(a^*) = \Phi(a)^*$ for all $a \in \mathcal{A}$. A $*$ -homomorphism which is bijective is then called a $*$ -isomorphism. (The inverse map is then easily seen to be a $*$ -isomorphism too.)

Exercise 48

Assume that Ω is a non-compact, locally compact Hausdorff space. Consider the subset of $C_b(\Omega)$ given by $\mathcal{A} := C_0(\Omega) + \mathbb{C}1_\Omega$.

- a) Check that $\mathcal{A}$ is a unital commutative C^* -subalgebra of $C_b(\Omega)$.
- b) Show that $\widehat{\mathcal{A}}$ may be identified with the one-point compactification $\Omega \cup \{\infty\}$ of Ω , and deduce that $\mathcal{A}$ is isometrically $*$ -isomorphic to $C(\Omega \cup \{\infty\})$.

Exercise 49

Let Ω be a topological space. We will denote the character space $\widehat{C_b(\Omega)}$ of $C_b(\Omega)$ by $\beta\Omega$. By Gelfand's theorem, we know that $\beta\Omega$ is a compact Hausdorff space.

- a) For $\omega \in \Omega$, define $\iota_\omega : C_b(\Omega) \rightarrow \mathbb{C}$ by $\iota_\omega(f) = f(\omega)$ for all $f \in C_b(\Omega)$. Check that $\iota_\omega \in \beta\Omega$. Then check that the map $\omega \mapsto \iota_\omega$ from Ω into $\beta\Omega$ is continuous.
- b) Let $h : \Omega \rightarrow \Omega'$ be a continuous map from Ω to some topological space Ω' . Define a map $\Phi_h : C_b(\Omega') \rightarrow C_b(\Omega)$ by

$$\Phi_h(g) = g \circ h$$

for all $g \in C_b(\Omega')$.

- i) Check that Φ_h is a bounded $*$ -homomorphism satisfying $\|\Phi_h\| = 1$.
 - ii) Check that Φ_h is isometric (and therefore injective) when h is surjective.
 - iii) Assume that Ω, Ω' are both compact Hausdorff spaces. Use Tietze's extension theorem (cf. Munkres' or Pedersen's book) to show that Φ_h is surjective whenever h is injective. Deduce that Φ_h is an isometric $*$ -isomorphism from $C(\Omega')$ onto $C(\Omega)$ whenever h is a homeomorphism.
- c) Assume $\Phi : C_b(\Omega') \rightarrow C_b(\Omega)$ is an (algebra) homomorphism such that $\Phi(1_{\Omega'}) = 1_\Omega$. Define $H_\Phi : \beta\Omega \rightarrow \beta\Omega'$ by

$$H_\Phi(\gamma) = \gamma \circ \Phi \quad \text{for all } \gamma \in \beta\Omega.$$

Check that H_Φ is well-defined and continuous.

- d) Use b) and c) to show that $\beta\Omega$ satisfies the following universal property:

For any continuous function $h : \Omega \rightarrow K$ from Ω into some compact Hausdorff space K , there exists a continuous function $\tilde{h} : \beta\Omega \rightarrow K$ such that

$$\tilde{h}(\iota_\omega) = h(\omega) \quad \text{for all } \omega \in \Omega.$$

Comment: If Ω is a so-called Tychonoff space (e.g., Ω is a locally compact Hausdorff space) then it may be shown that the map $\omega \mapsto \iota_\omega$ is a homeomorphism from Ω onto its range $\iota(\Omega)$. The space $\beta\Omega$ is then called the Stone-Cech compactification of Ω . You can read more about such spaces in Pedersen's book (cf. 4.3.17 and 4.3.18) if you are interested.

e) Assume that Ω, Ω' are both compact Hausdorff spaces. Show that the following assertions are equivalent:

- (i) Ω and Ω' are homeomorphic.
- (ii) $C(\Omega)$ and $C(\Omega')$ are isometrically $*$ -isomorphic (as C^* -algebras).
- (iii) $C(\Omega)$ and $C(\Omega')$ are isomorphic (as algebras).