

MAT4450 - Spring 2026 - Exercises - Set 2

Exercise 5

Let $\{x_\alpha\}_{\alpha \in \Lambda}$ be a net in a nonempty set X . Check that the following two claims hold (we used these during the lecture on Thursday):

a) Let C denote the collection of all families $\mathcal{F}$ of nonempty subsets of X satisfying that

- $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 \in \mathcal{F}$,
- $\{x_\alpha\}_{\alpha \in \Lambda}$ is frequently in F for every $F \in \mathcal{F}$.

As a partial order on C we use set-inclusion. Let $\{\mathcal{F}_j\}_{j \in J}$ be a totally ordered subset of C and set $\mathcal{G} := \bigcup_{j \in J} \mathcal{F}_j$. Then $\mathcal{G} \in C$.

(As every $\mathcal{F}_j$ is contained in $\mathcal{G}$, Zorn's lemma applies and gives that C has a maximal element).

b) If $A \subseteq X$ and $\{x_{\alpha_\beta}\}_{\beta \in \Lambda'}$ is a subnet of $\{x_\alpha\}_{\alpha \in \Lambda}$ that is frequently in A , then $\{x_\alpha\}_{\alpha \in \Lambda}$ is frequently in A .

Exercise 6

Assume X is a nonempty set with two topologies τ and τ' . Show that the following statements are equivalent:

- τ is weaker than τ' (equivalently, τ' is stronger than τ).
- Every net in X converging to some $x \in X$ w.r.t. τ' converges to x w.r.t. τ .
- $\overline{A}^{\tau'} \subseteq \overline{A}^\tau$ for every $A \subseteq X$.

Exercise 7

Let $\{X_i\}_{i \in I}$ be a family of topological spaces and consider $X := \prod_{i \in I} X_i$ with the product topology.

a) Show that X is Hausdorff if and only if all X_i 's are Hausdorff.

b) Let Y be a topological space and $f : Y \rightarrow X$ be a function. Show that f is continuous if and only if $p_i \circ f : Y \rightarrow X_i$ is continuous for each $i \in I$. (Here $p_i : X \rightarrow X_i$ denotes the canonical map given by $p_i(x) = x(i)$ for every $x \in X$).

Exercise 8

Let X be a nonempty set and let $\{\rho_i\}_{i \in I}$ be a family of functions from X to $\mathbb{R}$. Consider $\mathbb{R}$ as a topological space w.r.t. to its standard topology and let τ denote the weak topology on X determined by $\{\rho_i\}_{i \in I}$, i.e., τ is the topology on X generated by the family $\mathcal{F} = \{\rho_i^{-1}(U) : i \in I, U \text{ is an open subset of } \mathbb{R}\} \subseteq \mathcal{P}(X)$.

Let $x \in X$. For $i \in I$, $\varepsilon > 0$ and J a finite nonempty subset of I , set

$$V_{i,\varepsilon}(x) := \{y \in X : |\rho_i(y) - \rho_i(x)| < \varepsilon\} \subseteq X$$

$$\text{and } V_{J,\varepsilon}(x) := \bigcap_{i \in J} V_{i,\varepsilon}(x) = \{y \in X : |\rho_i(y) - \rho_i(x)| < \varepsilon \text{ for all } i \in J\} \subseteq X.$$

Moreover, set

$$\mathcal{V}_x := \left\{ V_{J,\varepsilon}(x) : \emptyset \neq J \subseteq I, J \text{ finite, and } \varepsilon > 0 \right\} \subseteq \mathcal{P}(X).$$

Show that $\mathcal{V}_x$ is a neighborhood basis at x (for τ).