

MAT4450 - Spring 2026 - Exercises - Set 3

Exercise 9

Let X be a vector space (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and let $\mathcal{S}$ be a family of seminorms on X . For each $\sigma \in \mathcal{S}$ and $z \in X$, define $\rho_{(\sigma,z)} : X \rightarrow \mathbb{R}^+$ by

$$\rho_{(\sigma,z)}(y) := \sigma(y - z) \quad \text{for all } y \in X.$$

The *weak topology on X induced by $\mathcal{S}$* , that we will denote by $\tau(\mathcal{S})$, is the weak topology on X determined by the family $\{\rho_{(\sigma,z)}\}_{(\sigma,z) \in \mathcal{S} \times X}$. Note that in general it is *not* the same as the weak topology determined by the family $\mathcal{S}$ itself. The purpose of this exercise is to describe a natural neighborhood basis at $x \in X$ for $\tau(\mathcal{S})$.

Let $x \in X$. For $\sigma \in \mathcal{S}$ and $\varepsilon > 0$, set $B_\varepsilon^\sigma(x) := \{y \in X : \sigma(y - x) < \varepsilon\}$.

Moreover, for $z \in X$, set $V_{(\sigma,z),\varepsilon}(x) := \{y \in X : |\sigma(y - z) - \sigma(x - z)| < \varepsilon\}$.

a) Check that for any $z \in X$ and $\varepsilon > 0$ we have

$$V_{(\sigma,x),\varepsilon}(x) = B_\varepsilon^\sigma(x) \subseteq V_{(\sigma,z),\varepsilon}(x).$$

b) If F is a finite nonempty subset of $\mathcal{S}$ and $\varepsilon > 0$, we set

$$B_\varepsilon^F(x) := \bigcap_{\sigma \in F} B_\varepsilon^\sigma(x) = \{y \in X : \sigma(y - x) < \varepsilon \text{ for all } \sigma \in F\}.$$

Use Exercise 8 and part a) to show that

$$\mathcal{B}_x := \{B_\varepsilon^F(x) : F \text{ is a finite nonempty subset of } \mathcal{S} \text{ and } \varepsilon > 0\}$$

is a neighborhood basis at x (for $\tau(\mathcal{S})$).

Exercise 10

Let (X, τ) be topological vector space (over $\mathbb{K}$) and let $\ell : X \rightarrow \mathbb{K}$ be a linear functional. Show that the following conditions are equivalent:

- a) ℓ is continuous on X .
- b) ℓ is continuous at some point of X .
- c) $\sup\{\operatorname{Re} \ell(u) \mid u \in U\} < \infty$ for some nonempty open $U \subseteq X$.
- d) $\inf\{\operatorname{Re} \ell(u) \mid u \in U\} > -\infty$ for some nonempty open $U \subseteq X$.
- e) $\sup\{|\ell(u)| \mid u \in U\} < \infty$ for some nonempty open $U \subseteq X$.

Exercise 11

Let X, X' be vector spaces (over $\mathbb{K}$). Let $\mathcal{S}$ (resp. $\mathcal{S}'$) denote a family of seminorms on X (resp. X') and let τ (resp. τ') denote the topology on X (resp. X') induced by $\mathcal{S}$ (resp. $\mathcal{S}'$), that is, $\tau = \tau(\mathcal{S})$ (resp. $\tau' = \tau(\mathcal{S}')$).

Let $T : X \rightarrow X'$ be a linear map, and consider T as map from (X, τ) to (X', τ') . Show that the following statements are equivalent:

- a) T is continuous.
- b) T is continuous at 0.
- c) For each $\sigma' \in \mathcal{S}'$ there exist a nonempty finite subset F of $\mathcal{S}$ and $M > 0$ such that

$$\sigma'(T(x)) \leq M \max_{\sigma \in F} \{\sigma(x)\} \quad \text{for all } x \in X.$$

Note: An immediate consequence is that a linear functional ℓ on X is continuous (w.r.t. τ) if and only if there exist a nonempty finite subset F of $\mathcal{S}$ and $M > 0$ such that

$$|\ell(x)| \leq M \max_{\sigma \in F} \{\sigma(x)\} \quad \text{for all } x \in X.$$

Exercise 12

Let X be a normed space (over $\mathbb{K}$).

- a) Assume X is finite-dimensional. Show that the weak topology on X agrees with the norm topology.
- b) Assume X is infinite-dimensional. Let U be a weakly open neighbourhood of 0 in X (i.e., $0 \in U \subseteq X$ and $U \in \tau_{\text{weak}}$). Show that U contains an infinite-dimensional subspace of X .
- c) Show that the weak topology on X does not coincide with the norm topology when X is infinite-dimensional.

Exercise 13

Let X be a normed space (over $\mathbb{K}$), $X \neq \{0\}$. We recall that if X is finite-dimensional, with $\dim(X) = n$, then the dual space X^* is finite-dimensional with $\dim(X^*) = n$. We also recall that if B denotes the closed unit ball in X , then B is compact (w.r.t. the norm-topology) if and only if X is finite-dimensional.

- a) Assume that X^* is finite-dimensional. Show that X is finite-dimensional too.
(Thus, we get that X is finite-dimensional $\Leftrightarrow X^*$ is finite-dimensional.)

We recall that the weak*-topology on X^* is the topology τ_{weak^*} determined by the family $\{j_x : x \in X\}$ of linear functionals on X^* defined for each $x \in X$ by $j_x(\varphi) = \varphi(x)$ for all $\varphi \in X^*$.

- b) Show that X is finite-dimensional if and only if the weak*-topology on X^* agrees with the norm-topology on X^* .
- c) Show that X is finite-dimensional if and only if the closed unit ball B^* in X^* is compact in the norm-topology on X^* .
- d) Assume that X is infinite-dimensional. Let W be a weak*-open neighborhood of 0 in X^* (i.e., $0 \in W \subseteq X^*$ and $W \in \tau_{\text{weak}^*}$). Show that W contains an infinite-dimensional subspace of X^* .