

MAT4450 - Spring 2026 - Exercises - Set 4

Exercise 14

Let X be a vector space (over $\mathbb{K}$) and let A be a nonempty subset of X . We recall that $a \in A$ is called an internal point of A if for all $x \in X \setminus \{0\}$ there exists some $\varepsilon > 0$ such that $a + \lambda x \in A$ for all $\lambda \in \mathbb{K}$ satisfying $|\lambda| < \varepsilon$. Note that it is not necessary to assume that A is convex for this definition to make sense. We let A^{int} denote the set of all internal points of A .

a) Assume that X is a topological vector space. Show that $A^o \subseteq A^{\text{int}}$, where A^o denotes the interior of A , i.e., the set of all interior points of A .¹

(Note: We have shown this fact in a lecture when A is convex and the topology on X is induced by a family of seminorms. Here we don't make these assumptions.)

b) Consider $X = \mathbb{R}^2$ with its usual topology and set

$$A = \{(x, y) \in [-1, 1] \times [-1, 1] : x^2 \leq y \text{ or } y \leq 0\} \subseteq \mathbb{R}^2.$$

Check that A is not convex and $A^o \neq A^{\text{int}}$.

c) Assume that X is a finite-dimensional normed space and A is convex. Show that $A^o = A^{\text{int}}$.

d) Consider $X = C([0, 1], \mathbb{R})$ as a normed space (over $\mathbb{R}$) w.r.t. the norm $\|f\|_1 = \int_0^1 |f(t)| dt$, and $A = \{f \in X \mid \sup_{t \in [0, 1]} |f(t)| < 1\}$. Check that A is convex and $A^o \neq A^{\text{int}}$.

Exercise 15

Let X be a nonzero vector space over $\mathbb{R}$. A subset H of X is called an *hyperplane* in X if there exists a nonzero linear functional $\ell : X \rightarrow \mathbb{R}$ and $c \in \mathbb{R}$ such that $H = \{x \in X \mid \ell(x) = c\}$, in which case the sets $\{x \in X \mid \ell(x) \geq c\}$ and $\{x \in X \mid \ell(x) \leq c\}$ are called the *half-spaces* determined by H .

Let $A, B \subseteq X$. We say that A and B can be *separated* if there exists a nonzero linear functional $\ell : X \rightarrow \mathbb{R}$ such that $\ell(x) \leq \ell(y)$ for all $x \in A$ and $y \in B$.

a) Check that A and B can be separated if and only if B and A can be separated, if and only if there exists a hyperplane in X such that A is contained in one of the half-spaces of H and B is contained in the other.

b) Check that the real version of the algebraic Hahn-Banach separation theorem can be formulated as follows:

Let A, B be disjoint nonempty convex subsets of X . If A or B contains an internal point, then A and B can be separated.

c) Consider $X := \{f : \mathbb{N} \rightarrow \mathbb{R} \mid f(j) = 0 \text{ for all } j\text{'s, except finitely many}\}$ as a vector space over $\mathbb{R}$ w.r.t. pointwise addition and scalar multiplication. For $f \in X \setminus \{0\}$, set

$$k_f := \min\{k \in \mathbb{N} : f(j) = 0 \text{ for all } j \geq k + 1\}.$$

Then define

$$A := \{f \in X \setminus \{0\} : f(k_f) > 0\} \quad \text{and} \quad B := \{f \in X \setminus \{0\} : f(k_f) < 0\}.$$

Check that A and B are nonempty convex subsets of X , none of which contains an internal point. Then show that A and B can not be separated.

This illustrates that the assumption in b) that A or B contains an internal point can not be skipped in general.

¹A point $x \in A$ is called an interior point of A if A is a neighbourhood of x .

Exercise 16

Give an example of two nonempty closed convex subsets K and F of $\mathbb{R}^2$ such that the conclusion of the Hahn-Banach separation theorem nr. III does not hold, that is, there is no linear functional φ on $\mathbb{R}^2$ such that

$$\sup\{\varphi(\mathbf{x}) : \mathbf{x} \in K\} < \inf\{\varphi(\mathbf{x}) : \mathbf{x} \in F\}.$$

Note that F and K must both be unbounded; otherwise, one of these would be bounded, hence compact, and the Hahn-Banach separation theorem nr. III would apply.

Exercise 17

Exercise 2.4.6 in Pedersen's book.