

MAT4450 - Spring 2026 - Exercises - Set 5

Exercise 18

Let X, Y, Z be normed spaces (over $\mathbb{K}$).

- a) Show that the map $T \mapsto T^\sharp$ from $\mathcal{B}(X, Y)$ to $\mathcal{B}(Y^*, X^*)$ is linear.
- b) Let $T \in \mathcal{B}(X, Y)$ and $S \in \mathcal{B}(Y, Z)$. Check that $(S \circ T)^\sharp = T^\sharp \circ S^\sharp$.

Exercise 19

- a) Let $n, m \in \mathbb{N}$, and choose some norms on $\mathbb{K}^n$ and $\mathbb{K}^m$. (Since all norms on a finite dimensional vector space are equivalent, it does not matter which norms you choose). Let A be some $m \times n$ matrix over $\mathbb{K}$, and $T : \mathbb{K}^n \rightarrow \mathbb{K}^m$ be the linear map having A as its standard matrix. Recall that T is automatically bounded.

Identify $\mathbb{K}^n$ with $(\mathbb{K}^n)^*$ via the isomorphism $y \mapsto \ell_y$, where $\ell_y(x) := x \cdot y = \sum_{i=1}^n x_i y_i$ for $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{K}^n$. Similarly, identify $\mathbb{K}^m$ with $(\mathbb{K}^m)^*$.

Check that the standard matrix of the adjoint operator $T^\sharp : \mathbb{K}^m \rightarrow \mathbb{K}^n$ is the transpose of A .

- b) Let X be a normed space over $\mathbb{K}$, and let $\varphi \in X^*$, i.e., $\varphi \in \mathcal{B}(X, \mathbb{K})$. Identify $\mathbb{K}$ with $(\mathbb{K})^*$ in the obvious way.

Check that the adjoint map $\varphi^\sharp \in \mathcal{B}(\mathbb{K}, X^*)$ is given by $\varphi^\sharp(\lambda) = \lambda\varphi$ for all $\lambda \in \mathbb{K}$.

Exercise 20

Consider $X = \ell^1(\mathbb{N}, \mathbb{K})$ as a normed space w.r.t. the $\|\cdot\|_1$ -norm. We recall that $\ell^\infty(\mathbb{N}, \mathbb{K})$, equipped with the $\|\cdot\|_\infty$ -norm, may be identified with X^* via the isometric isomorphism $g \mapsto \varphi_g$, where

$$\varphi_g(f) = \sum_{n=1}^{\infty} f(n)g(n)$$

for all $f \in \ell^1(\mathbb{N}, \mathbb{K})$, $g \in \ell^\infty(\mathbb{N}, \mathbb{K})$.

Set $Y := c_0(\mathbb{N}, \mathbb{K}) = \{g \in \ell^\infty(\mathbb{N}, \mathbb{K}) \mid \lim_{n \rightarrow \infty} g(n) = 0\}$, and equip Y with the $\|\cdot\|_\infty$ -norm. We also recall that X may be identified with Y^* , via the isometric isomorphism $f \mapsto \psi_f$, where

$$\psi_f(g) = \sum_{n=1}^{\infty} f(n)g(n)$$

for all $f \in X = \ell^1(\mathbb{N}, \mathbb{K})$, $g \in Y = c_0(\mathbb{N}, \mathbb{K})$.

- a) Define a linear map $T : X \rightarrow Y$ by

$$[T(f)](n) = \sum_{m=n}^{\infty} f(m)$$

for all $f \in X$ and $n \in \mathbb{N}$. Check that T is bounded. Then find an expression for $T^\sharp \in \mathcal{B}(Y^*, X^*) = \mathcal{B}(X, X^*)$.

- b) Consider Y as a subspace of X^* . Check that Y is norm-closed in X^* . Then show that $(Y_\perp)^\perp = X^*$, and conclude that $Y \neq (Y_\perp)^\perp$.

Exercise 21

Let X and Y be Banach spaces (over $\mathbb{K}$), and assume $T \in \mathcal{B}(X, Y)$ is such that its range $T(X)$ is closed in Y . Let $T^\sharp : Y^* \rightarrow X^*$ denote the adjoint of T .

Show that $T^\sharp(Y^*) = (\ker T)^\perp$.

(This is Exercise 2.3.8 in Pedersen's book. You will find a hint there if you need it).

Exercise 22

Let X be a normed space (over $\mathbb{K}$) and N be a norm-closed subspace of X^* .

Show that N is separating for X if and only if N is weak*-dense in X^* .

Exercise 23

Let H be a (complex) Hilbert space and M be a closed subspace of H . If we consider H as a normed space, then the annihilator of M is given by

$$M^\perp = \{\varphi \in H^* : \varphi(x) = 0 \text{ for all } x \in M\}.$$

On the other hand, let $M^{\perp_H}$ denote the orthogonal complement of M in H , i.e.,

$$M^{\perp_H} = \{y \in H : \langle x, y \rangle = 0 \text{ for all } x \in M\}.$$

a) Recall that one may identify H with H^* via the conjugate linear surjective isometric map $\Phi : H \rightarrow H^*$ given by $\Phi(y) = \varphi_y$ for each $y \in H$, where $\varphi_y(x) = \langle x, y \rangle$ for all $x \in H$.

Check that $M^\perp = \Phi(M^{\perp_H})$.

(We may therefore identify $M^\perp$ with $M^{\perp_H}$ via Φ , and one usually write $M^\perp$ for both.)

b) Consider the quotient space H/M with the quotient norm. Check that H/M is isometrically isomorphic with $M^{\perp_H}$.

c) Let $T \in \mathcal{B}(H)$. Recall that $T^* \in \mathcal{B}(H)$ denotes the Hilbert space adjoint of T .

Show that $T^\sharp = \Phi T^* \Phi^{-1}$.

Exercise 24

Let X be a normed space (over $\mathbb{K}$) and M be a norm-closed subspace of X . Let $Q : X \rightarrow X/M$ denote the quotient map (so Q is linear and contractive).

Show that the adjoint map $Q^\sharp : (X/M)^* \rightarrow X^*$ is isometric.