

MAT4450 - Spring 2026 - Exercises - Set 6

Exercise 25

Let Ω be a compact Hausdorff space. Assume $\{f_n\}$ is a sequence in $C(\Omega, \mathbb{K})$ such that $\sup_{n \in \mathbb{N}} \|f_n\|_\infty < \infty$ and $\{f_n\}$ converges to some $f \in C(\Omega, \mathbb{K})$ pointwise on Ω . Show that

$$f \in \overline{\text{co}(\{f_n, n \in \mathbb{N}\})}^{\|\cdot\|_\infty}.$$

Exercise 26

Let X be a vector space (over $\mathbb{K}$).

a) Let A be a nonempty subset of X and set

$$\text{co}(A) := \left\{ \sum_{j=1}^n \lambda_j a_j \mid n \in \mathbb{N}, \lambda_1, \dots, \lambda_n \in [0, 1], \sum_{j=1}^n \lambda_j = 1, a_1, \dots, a_n \in A \right\}.$$

Show that $\text{co}(A)$, the *convex hull* of A , is the least convex subset of X containing A .

b) Let $C \neq \emptyset$ be a convex subset of X . Assume F is a face of C , and K is a face of F . Show that K is a face of C .

Exercise 27

Let (X, τ) be a locally convex Hausdorff topological vector space and K be a nonempty compact convex subset of X .

Let $\varphi \in (X, \tau)^*$ and set

$$m := \inf\{\text{Re } \varphi(x) : x \in K\}, M := \sup\{\text{Re } \varphi(x) : x \in K\}, s := \sup\{|\varphi(x)| : x \in K\}.$$

Show that there exist $x, y, z \in \text{ex}(K)$ such that

$$\text{Re } \varphi(x) = m, \text{Re } \varphi(y) = M, |\varphi(z)| = s.$$

Exercise 28

We recall that if C is a nonempty convex subset of a vector space, then $\text{ex}(C)$ denotes the set of all extreme points of C .

Let H be a complex Hilbert space $\neq \{0\}$. Set

$$B := \{\xi \in H : \|\xi\| \leq 1\} \text{ and } \mathcal{B} := \{T \in \mathcal{B}(H) : \|T\| \leq 1\}.$$

a) Show that $\text{ex}(B) = \{\eta \in B : \|\eta\| = 1\}$.

b) Let $T \in \mathcal{B}(H)$. Assume that T or T^* is isometric.¹ Show that $T \in \text{ex}(\mathcal{B})$.

Exercise 29

Consider the Banach space $X = (L^1(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu), \|\cdot\|_1)$ (over $\mathbb{K}$), where $\mathcal{B}_{\mathbb{R}}$ denotes the σ -algebra of all Borel subsets of $\mathbb{R}$ and μ denotes the Lebesgue measure on $\mathcal{B}_{\mathbb{R}}$.

Consider the closed ball $B := \{f \in X : \|f\|_1 \leq 1\}$, which is clearly convex.

a) Show that B has no extreme points.

Hint. Assume for contradiction that there exists some $f \in \text{ex}(B)$. Show first that $\|f\|_1 = 1$. Then consider the continuous function $F(t) := \int_{(-\infty, t]} |f| d\mu$, $t \in \mathbb{R}$.

b) Deduce that there is no topology on X making it a locally convex Hausdorff topological vector space such that B is compact. Explain why this implies that $(L^1(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu), \|\cdot\|_1)$ can not be isomorphic to the dual space of $(L^\infty(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu), \|\cdot\|_\infty)$.

¹By T^* we mean here the (Hilbert) adjoint operator of T , as defined for a bounded operator on a Hilbert space.