

MAT4450 - Spring 2026 - Exercises - Set 7

Exercise 30

Let Ω be a nonempty compact subset of $\mathbb{R}^2$ and consider the set $\mathcal{P}$ of all real polynomials in two commuting variables as a subset of $C(\Omega, \mathbb{R})$. Show that $\mathcal{P}$ is dense in $C(\Omega, \mathbb{R})$ (w.r.t. $\|\cdot\|_\infty$).

Exercise 31

Let Ω be a compact Hausdorff space, and let $\mathcal{A}$ be a subalgebra of $C(\Omega, \mathbb{R})$.

- a) Show that $|f|^{1/n} \in \overline{\mathcal{A}}^{\|\cdot\|_\infty}$ for every $f \in \mathcal{A}$ and $n \in \mathbb{N}$.
- b) Assume that $\mathcal{A}$ separates the points of Ω and that there exists some $g \in \mathcal{A}$ such that $g(x) \neq 0$ for all $x \in \Omega$. Show that $\overline{\mathcal{A}}^{\|\cdot\|_\infty} = C(\Omega, \mathbb{R})$.

Exercise 32.

Let H denote a nontrivial Hilbert space. Recall that $\mathcal{K}(H)$ denotes the set of all compact linear operators on H , and that it is a Banach algebra (w.r.t. operator norm).

Show that $\mathcal{K}(H)$ is unital if and only if H is finite-dimensional.

Exercise 33.

We recall that if Ω is a topological space, then

$$C_b(\Omega, \mathbb{K}) = \{f : \Omega \rightarrow \mathbb{K} : f \text{ is continuous and bounded}\}$$

is a commutative Banach algebra over $\mathbb{K}$ (w.r.t. $\|\cdot\|_\infty$).

- a) Consider the set

$$C_0(\mathbb{R}, \mathbb{K}) := \{f : \mathbb{R} \rightarrow \mathbb{K} \mid f \text{ is continuous and } \lim_{t \rightarrow \infty} f(t) = \lim_{t \rightarrow -\infty} f(t) = 0\}$$

Show that $C_0(\mathbb{R}, \mathbb{K})$ is a norm-closed subalgebra of $C_b(\mathbb{R}, \mathbb{K})$, hence that $C_0(\mathbb{R}, \mathbb{K})$ is a commutative Banach algebra (w.r.t. $\|\cdot\|_\infty$). Check also that $C_0(\mathbb{R}, \mathbb{K})$ is non-unital.

- b) More generally, let Ω denote a locally compact Hausdorff space (e.g. $\Omega = \mathbb{R}^n$ with its standard topology). We say that a function $f : \Omega \rightarrow \mathbb{K}$ *vanishes at infinity* if for every $\varepsilon > 0$ the set $\{x \in \Omega : |f(x)| \geq \varepsilon\}$ is compact.¹ Set

$$C_0(\Omega, \mathbb{K}) := \{f \in C(\Omega, \mathbb{K}) \mid f \text{ vanishes at infinity}\}.$$

Show that $C_0(\Omega, \mathbb{K})$ is a norm-closed subalgebra of $C_b(\Omega, \mathbb{K})$; hence, $C_0(\Omega, \mathbb{K})$ is a commutative Banach algebra (w.r.t. $\|\cdot\|_\infty$). Then show that $C_0(\Omega, \mathbb{K})$ is unital if and only if Ω is compact.

¹You should check that if $\Omega = \mathbb{R}$ with its standard topology, then $f \in C(\mathbb{R}, \mathbb{K})$ vanishes at infinity if and only if $\lim_{t \rightarrow \pm\infty} f(t) = 0$.