

MAT4450 - Spring 2026 - Exercises - Set 8

Exercise 34

Let $\mathcal{A}$ be a non-unital Banach algebra over $\mathbb{K}$. Set $\tilde{\mathcal{A}} = \{(a, \alpha) \mid a \in \mathcal{A}, \alpha \in \mathbb{K}\}$ and define addition, multiplication by scalars and product by

$$\begin{aligned}(a, \alpha) + (b, \beta) &= (a + b, \alpha + \beta), \\ \lambda(a, \alpha) &= (\lambda a, \lambda \alpha), \\ (a, \alpha)(b, \beta) &= (ab + \alpha b + \beta a, \alpha \beta)\end{aligned}$$

for $(a, \alpha), (b, \beta) \in \tilde{\mathcal{A}}$ and $\lambda \in \mathbb{K}$. Check for yourself that $\tilde{\mathcal{A}}$ becomes a unital algebra with unit $e := (0, 1)$, and that the map $(a, \alpha) \mapsto \alpha$ is an algebra-homomorphism from $\tilde{\mathcal{A}}$ into $\mathbb{K}$, with kernel equal to $\mathcal{A}_0 := \{(a, 0) \mid a \in \mathcal{A}\}$. It is usual to identify $\mathcal{A}$ with the ideal $\mathcal{A}_0$ and to write (a, α) as $a + \alpha e$ (since $(a, \alpha) = (a, 0) + \alpha(0, 1)$).

Define $\|\cdot\|$ on $\tilde{\mathcal{A}}$ by

$$\|a + \alpha e\| := \|a\| + |\alpha|$$

for every $a + \alpha e \in \tilde{\mathcal{A}}$.

Verify that $\|\cdot\|$ is a norm on $\tilde{\mathcal{A}}$, and that $\tilde{\mathcal{A}}$ becomes a unital Banach algebra, containing $\mathcal{A}$ as a closed ideal. Check also that the quotient $\tilde{\mathcal{A}}/\mathcal{A}$ is isometrically isomorphic to $\mathbb{K}$.

Exercise 35

Consider the commutative Banach algebra $\mathcal{A} = C(\Omega, \mathbb{K})$, where Ω is a compact Hausdorff space. Let $\omega_0 \in \Omega$ and define $\varphi : \mathcal{A} \rightarrow \mathbb{K}$ by $\varphi(f) = f(\omega_0)$ for $f \in \mathcal{A}$.

a) Check that φ is a continuous algebra-homomorphism from $\mathcal{A}$ into $\mathbb{K}$ (considered as a Banach algebra) satisfying $\|\varphi\| = 1$.

b) Consider the closed ideal of $\mathcal{A}$ given by $\mathcal{I} = \ker \varphi$. Show that the Banach algebra $\mathcal{A}/\mathcal{I}$ is isometrically isomorphic to $\mathbb{K}$.

Note: From now on, we use the convention that by a unital Banach algebra $\mathcal{A}$ we always mean a Banach algebra over $\mathbb{K}$ having a unit e such that $\|e\| = 1$.

Exercise 36

Let $\mathcal{A}$ be a unital Banach algebra. Recall that $\text{GL}(\mathcal{A}) = \{a \in \mathcal{A} : a \text{ is invertible in } \mathcal{A}\}$.

a) Let $a \in \text{GL}(\mathcal{A})$ and note that $0 \notin \text{sp}_{\mathcal{A}}(a)$. Show that

$$\text{sp}_{\mathcal{A}}(a^{-1}) = \left\{ \lambda^{-1} \mid \lambda \in \text{sp}_{\mathcal{A}}(a) \right\}.$$

b) Let $\mathcal{B}$ be unital Banach algebra and assume that $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is an algebra-isomorphism. Show that $\text{sp}_{\mathcal{A}}(a) = \text{sp}_{\mathcal{B}}(\phi(a))$ for every $a \in \mathcal{A}$.

Exercise 37 (= Exercise 4.1.3 in Pedersen's book)

Let a and b be elements in a complex, unital Banach algebra $\mathcal{A}$. Show that

$$\text{sp}_{\mathcal{A}}(ab) \setminus \{0\} = \text{sp}_{\mathcal{A}}(ba) \setminus \{0\}.$$

Exercise 38

Consider the complex Hilbert space $H = L^2([0, 1], \mathcal{B}_{[0,1]}, \mu)$, where μ denotes the Lebesgue measure on the Borel σ -algebra $\mathcal{B}_{[0,1]}$.

Set $\mathcal{A} := \mathcal{B}(H)$, and let $M \in \mathcal{A}$ denote the multiplication operator given by

$$[M(g)](t) = tg(t) \quad \text{for all } g \in H \text{ and } t \in [0, 1].$$

As we have seen in a previous course, M has no eigenvalues.

Show that $\text{sp}_{\mathcal{A}}(M) = [0, 1]$.

Exercise 39

Let Ω be a topological space and $\mathcal{A} = C_b(\Omega)$, i.e., $\mathcal{A}$ is the unital Banach algebra consisting of all bounded continuous complex functions on Ω , equipped with the uniform norm $\|\cdot\|_{\infty}$.

Let $f \in \mathcal{A}$. Show that $\text{sp}_{\mathcal{A}}(f) = \overline{f(S)}$.

Note: If S is a set, we can consider it as a topological space w.r.t. the discrete topology, and we then have $C_b(S) = \ell^{\infty}(S)$. Thus the formula $\text{sp}_{\mathcal{A}}(f) = \overline{f(S)}$ holds for all $f \in \mathcal{A} = \ell^{\infty}(S)$.

Exercise 40

Let X be a complex Banach space and consider the unital Banach algebra $\mathcal{A} := \mathcal{B}(X)$. Let $T \in \mathcal{A}$. Recall that the adjoint operator $T^{\sharp}$ belongs then to the unital Banach algebra $\mathcal{B} := \mathcal{B}(X^*)$.

Show that $\text{sp}_{\mathcal{B}}(T^{\sharp}) = \text{sp}_{\mathcal{A}}(T)$.

Note that if H is a complex Hilbert space and $T \in \mathcal{A} := \mathcal{B}(H)$, then it is very easy to show that $\text{sp}_{\mathcal{A}}(T^*) = \{\bar{\lambda} : \lambda \in \text{sp}_{\mathcal{A}}(T)\}$. You are invited to do this as a warm-up. Note further that one inclusion in the equality above can be shown by elementary means. On the other hand, the other inclusion is quite challenging to prove. You can ask for a hint if needed!