

27th April, 2026

MAT4450 – Spring 2025

Mandatory assignment - Solution outline

Problem 1

Let M be a closed subspace of a locally convex Hausdorff space (X, τ) (over $\mathbb{K}$) and let $\psi : M \rightarrow \mathbb{K}$ be a continuous linear functional.

Show that ψ can be extended to some $\varphi \in (X, \tau)^*$.

Solution: We follow the given hint. If $\psi = 0$, the assertion is trivial. So we may assume that $\psi \neq 0$. We may then pick $y_0 \in M$ such that $\psi(y_0) \neq 0$. Setting $x_0 = \frac{1}{\psi(y_0)} y_0$, we have $x_0 \in M$ and $\psi(x_0) = 1$.

Set $N := \{x \in M : \psi(x) = 0\}$. Then N is a subspace of M , hence of X . Moreover, N is closed in X (since $N = \ker(\psi)$ is closed in M).

As $\psi(x_0) \neq 0$, we have $x_0 \in X \setminus N$. Hence a corollary to one of the Hahn-Banach separation theorems (nr. II in the lectures) gives that there exists $\varphi' \in (X, \tau)^*$ such that $\varphi' \in N^\perp$, i.e., $\varphi' = 0$ on N , and $\varphi'(x_0) \neq 0$.

Set $\varphi := \frac{1}{\varphi'(x_0)} \varphi'$. Then $\varphi \in (X, \tau)^*$, $\varphi(x_0) = 1$ and $\varphi = 0$ on N .

Now, consider $x \in M$. Then

$$\psi(x - \psi(x)x_0) = \psi(x) - \psi(x)\psi(x_0) = \psi(x) - \psi(x) = 0,$$

so $x - \psi(x)x_0 \in N$. As $x = (x - \psi(x)x_0) + \psi(x)x_0$ and $\varphi|_N = 0$, we get that

$$\varphi(x) = \varphi(\psi(x)x_0) = \psi(x)\varphi(x_0) = \psi(x).$$

Thus, φ agrees with ψ on M , as desired.

Problem 2

Let Ω be a compact Hausdorff space. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence in $C(\Omega, \mathbb{K})$ and $f \in C(\Omega, \mathbb{K})$. Assume that $\sup_{n \in \mathbb{N}} \|f_n\|_\infty < \infty$ and $\{f_n\}_{n \in \mathbb{N}}$ converges to f pointwise on Ω .

a) Give an example showing that $\{f_n\}_{n \in \mathbb{N}}$ does not necessarily converge to f uniformly.

Let $\Omega = [0, 1]$. For each $n \in \mathbb{N}$, let $f_n : [0, 1] \rightarrow \mathbb{R}$ be a continuous function taking all its values in $[0, 1]$, which is zero outside the interval $(1/(n+1), 1/n)$ and takes the value 1 at the midpoint of $(1/(n+1), 1/n)$. Then $\|f_n\|_\infty = 1$ for all $n \in \mathbb{N}$ and $f_n \rightarrow 0$ pointwise on Ω as $n \rightarrow \infty$. However, $\{f_n\}_{n \in \mathbb{N}}$ does not converge to 0 uniformly on Ω .

b) Show that

$$f \in \overline{\text{co}(\{f_n, n \in \mathbb{N}\})}^{\|\cdot\|_\infty}.$$

To prove this assertion, we observe that, since $\text{co}(\{f_n, n \in \mathbb{N}\})$ is a convex subset of $C(\Omega, \mathbb{K})$, we know from a result proved in the course that

$$\overline{\text{co}(\{f_n, n \in \mathbb{N}\})}^{\|\cdot\|_\infty} = \overline{\text{co}(\{f_n, n \in \mathbb{N}\})}^{\text{weak}}.$$

It therefore suffices to show that $f_n \rightarrow f$ as $n \rightarrow \infty$ in the weak topology. Let us assume that $\mathbb{K} = \mathbb{C}$. (The case where $\mathbb{K} = \mathbb{R}$ can be handled in a similar way.)

We recall that the dual space of $C(\Omega) = C(\Omega, \mathbb{C})$ may be identified with the space $M(\Omega)$ of all regular complex Borel measures on Ω . If $\mu \in M(\Omega) = C(\Omega)^*$ and $f \in C(\Omega)$, this identification means that we have $\mu(f) = \int_X f d\mu$. A natural idea to prove the desired statement is therefore to invoke Lebesgue's Dominated Convergence Theorem (LDCT).

Consider first $\mu \in M(\Omega)^+$. Set $s := \sup_{n \in \mathbb{N}} \|f_n\|_\infty$ (which is a finite number by assumption). Then we have that $|f_n(x)| \leq s$ for all $x \in \Omega$, so $|f_n| \leq s 1_\Omega$ on Ω for all $n \in \mathbb{N}$. Since μ is a finite measure, the function $s 1_\Omega$ is integrable w.r.t. to μ . Hence we may apply the LDCT and get that every f_n is integrable w.r.t. μ , and

$$\lim_{n \rightarrow \infty} \mu(f_n) = \lim_{n \rightarrow \infty} \int_\Omega f_n d\mu = \int_\Omega f d\mu = \mu(f).$$

Now, since any $\mu \in M(\Omega)$ is a linear combination of measures in $M(\Omega)^+$, we get that $\lim_{n \rightarrow \infty} \mu(f_n) = \mu(f)$ for all $\mu \in M(\Omega) = C(\Omega)^*$, i.e., $f_n \rightarrow f$ weakly, as desired.

Problem 3

Let X and Y be compact Hausdorff spaces. Set

$$\mathcal{A} := \text{span}\left(\{f \otimes g : f \in C(X), g \in C(Y)\}\right) \subseteq C(X \times Y),$$

where $f \otimes g \in C(X \times Y)$ denotes the function on $X \times Y$ defined by

$$(f \otimes g)(x, y) := f(x)g(y) \text{ for all } (x, y) \in X \times Y.$$

Show that $\mathcal{A}$ is dense in $C(X \times Y)$ (w.r.t. the uniform norm).

Solution. It is straightforward to verify that $\mathcal{A}$ is a self-adjoint subalgebra of $C(X \times Y)$ which contains the constant functions and separates the points of $X \times Y$. We skip the details here. It therefore follows from the (complex) Stone-Weierstrass theorem that $\mathcal{A}$ is dense in $C(X \times Y)$ (w.r.t. the uniform norm).

Problem 4

Let $\mathcal{A}$ be a unital complex Banach algebra. Let $a \in \mathcal{A}$ and U be an open subset of $\mathbb{C}$ containing $\text{sp}(a)$.

Show that there exists $\varepsilon > 0$ such that

$$b \in \mathcal{A} \text{ and } \|a - b\| < \varepsilon \Rightarrow \text{sp}(b) \subseteq U.$$

Solution. Assume (for contradiction) that the assertion does not hold. For every $n \in \mathbb{N}$ we can then find some $b_n \in \mathcal{A}$ such that $\|a - b_n\| < n^{-1}$ and $\text{sp}(b_n)$ is not contained in U , i.e., $\text{sp}(b_n) \setminus U \neq \emptyset$. We may then pick $\lambda_n \in \text{sp}(b_n) \setminus U$ for every $n \in \mathbb{N}$. Then we have

$$|\lambda_n| \leq \|b_n\| \leq \|b_n - a\| + \|a\| \leq \frac{1}{n} + \|a\| \leq \|a\| + 1$$

for every n , so the sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ is bounded in $\mathbb{C}$. By the Bolzano-Weierstrass theorem, this sequence has a convergent subsequence. Hence, by passing to a subsequence if necessary, we may assume that $\{\lambda_n\}_{n \in \mathbb{N}}$ is convergent, say $\lim_{n \rightarrow \infty} \lambda_n = \lambda$ for some $\lambda \in \mathbb{C}$.

Since $\{\lambda_n\}_{n \in \mathbb{N}} \subseteq \mathbb{C} \setminus U$ and $\mathbb{C} \setminus U$ is closed, we get that $\lambda \in \mathbb{C} \setminus U$. Since $\text{sp}(a) \subseteq U$ (by assumption), this implies that $\lambda \notin \text{sp}(a)$, i.e., $\lambda e - a \in \text{GL}(\mathcal{A})$, where e denotes the unit of $\mathcal{A}$. But $\lambda_n e - b_n$ converges to $\lambda e - a$, and $\text{GL}(\mathcal{A})$ is an open subset of $\mathcal{A}$, so we can find some $N \in \mathbb{N}$ such that $\lambda_N e - b_N \in \text{GL}(\mathcal{A})$, i.e., $\lambda_N \notin \text{sp}(b_N)$. As $\lambda_N \in \text{sp}(b_N)$, this gives a contradiction.

Problem 5

Let $\mathcal{A}$ be a *commutative* unital Banach algebra. Define the *radical* of $\mathcal{A}$ by

$$\text{rad}(\mathcal{A}) = \bigcap_{\gamma \in \widehat{\mathcal{A}}} \ker(\gamma).$$

We will say that $\mathcal{A}$ is *semisimple* when $\text{rad}(\mathcal{A}) = \{0\}$.

a) Let Ω be a topological space. Check that $C_b(\Omega)$ is semisimple.

Solution. For every $\omega \in \Omega$ let $\gamma_\omega : C_b(\Omega) \rightarrow \mathbb{C}$ be defined by

$$\gamma_\omega(f) = f(\omega) \quad \text{for all } f \in C_b(\Omega).$$

Then it is straightforward to check that each γ_ω is a character on $C_b(\Omega)$. Thus, we get

$$\begin{aligned} \text{rad}(\mathcal{A}) &\subseteq \bigcap_{\omega \in \Omega} \ker(\gamma_\omega) = \{ f \in C_b(\Omega) : \gamma_\omega(f) = 0 \text{ for all } \omega \in \Omega \} \\ &= \{ f \in C_b(\Omega) : f(\omega) = 0 \text{ for all } \omega \in \Omega \} = \{0\}, \end{aligned}$$

hence $\text{rad}(\mathcal{A}) = \{0\}$.

b) Show that $\mathcal{A}$ is semisimple if and only if it holds that $a = 0$ whenever $a \in \mathcal{A}$ and $r_{\mathcal{A}}(a) = 0$.

Solution. Assume $\mathcal{A}$ is semisimple, and consider $a \in \mathcal{A}$ such that $r_{\mathcal{A}}(a) = 0$. For every $\gamma \in \widehat{\mathcal{A}}$, we know that $\gamma(a) \in \text{sp}(a)$, hence $|\gamma(a)| \leq r_{\mathcal{A}}(a) = 0$. Thus

$\gamma(a) = 0$ for every $\gamma \in \widehat{\mathcal{A}}$, i.e., $a \in \text{rad}(\mathcal{A})$. Since $\text{rad}(\mathcal{A}) = \{0\}$ by assumption, this implies that $a = 0$. This shows that the RHS of the assertion above holds.

Conversely, assume that the RHS of the assertion above holds, and let $a \in \text{rad}(\mathcal{A})$. Then $\gamma(a) = 0$ for every $\gamma \in \widehat{\mathcal{A}}$. This gives that

$$[\Gamma_{\mathcal{A}}(a)](\gamma) = \gamma(a) = 0 \text{ for all } \gamma \in \widehat{\mathcal{A}},$$

hence that $\Gamma_{\mathcal{A}}(a) = 0$. Thus $r_{\mathcal{A}}(a) = \|\Gamma_{\mathcal{A}}(a)\|_{\infty} = 0$. Using the assumption, this implies that $a = 0$. This shows that $\text{rad}(\mathcal{A}) = \{0\}$, i.e., $\mathcal{A}$ is semisimple.

c) Let $\mathcal{A}$ be a semisimple commutative unital Banach algebra and $\mathcal{B}$ be unital Banach algebra. Assume $\Phi : \mathcal{B} \rightarrow \mathcal{A}$ is an algebra-homomorphism mapping the unit of $\mathcal{B}$ to the unit of $\mathcal{A}$. Show that Φ is norm-continuous.

Solution. To show that Φ is norm-continuous, it suffices to show that its graph is closed. Assume therefore that $\{b_n\}$ is a sequence in $\mathcal{B}$ which converges in norm to $b \in \mathcal{B}$ and is such that $\{\Phi(b_n)\}$ converges in norm to $a \in \mathcal{A}$. We have to show that $\Phi(b) = a$.

Let $\gamma \in \widehat{\mathcal{A}}$. Then $\gamma \circ \Phi : \mathcal{B} \rightarrow \mathbb{C}$ is an algebra-homomorphism such that $(\gamma \circ \Phi)(e_{\mathcal{B}}) = \gamma(e_{\mathcal{A}}) = 1$, i.e., $\gamma \circ \Phi \in \widehat{\mathcal{B}}$. Thus γ and $\gamma \circ \Phi$ are both norm-continuous, so we get that

$$\begin{aligned} \gamma(a) &= \gamma\left(\lim_{n \rightarrow \infty} \Phi(b_n)\right) = \lim_{n \rightarrow \infty} \gamma(\Phi(b_n)) = \lim_{n \rightarrow \infty} (\gamma \circ \Phi)(b_n) \\ &= (\gamma \circ \Phi)\left(\lim_{n \rightarrow \infty} b_n\right) = (\gamma \circ \Phi)(b) = \gamma(\Phi(b)). \end{aligned}$$

Thus, we have $\gamma(a - \Phi(b)) = 0$ for all $\gamma \in \widehat{\mathcal{A}}$, that is, $a - \Phi(b) \in \text{rad}(\mathcal{A})$. Since $\text{rad}(\mathcal{A}) = \{0\}$ (by assumption), this implies that $a - \Phi(b) = 0$, as desired.

d) Deduce that on a semisimple commutative unital Banach algebra all Banach algebra norms are equivalent.

Solution. Let $\mathcal{A}$ be a semisimple commutative unital Banach algebra w.r.t. a norm $\|\cdot\|$, and assume $\|\cdot\|'$ is also a norm on $\mathcal{A}$ making it into a Banach algebra. Let $\text{id} : \mathcal{A} \rightarrow \mathcal{A}$ denote the identity map. Then id is an algebra-homomorphism mapping the unit of $\mathcal{A}$ to itself.

Considering id as map from $(\mathcal{A}, \|\cdot\|')$ to $(\mathcal{A}, \|\cdot\|)$, part c) gives that id is norm-continuous, hence bounded. Thus there is some $M_1 > 0$ such that $\|a\| \leq M_1 \|a\|'$ for all $a \in \mathcal{A}$. Moreover, by the bounded inverse theorem, we get that the inverse map id^{-1} is also norm-bounded, i.e., there is some $M_2 > 0$ such that $\|a\|' \leq M_2 \|a\|$ for all $a \in \mathcal{A}$. This shows that $\|\cdot\|$ and $\|\cdot\|'$ are equivalent.

Problem 6

Let $N \in \mathbb{N}$ and let $\mathbb{Z}_N = \{0, 1, \dots, N-1\}$ denote the cyclic group of order N (addition being defined modulo N). Set $\mathcal{A} = \ell^1(\mathbb{Z}_N, \mathbb{C})$. It can easily be checked that $\mathcal{A}$ becomes a commutative unital Banach algebra w.r.t. the convolution product given by

$$(f * g)(n) = \sum_{m \in \mathbb{Z}_N} f(m)g(n-m) \quad \text{for } f, g \in \mathcal{A}, n \in \mathbb{Z}_N.$$

Explain that the character space $\widehat{\mathcal{A}}$ of $\mathcal{A}$ can be identified with $\mathbb{Z}_N$.

Solution. We can proceed essentially in the same way as when dealing with the infinite cyclic group $\mathbb{Z}$, but there is an important difference: we get here that $\widehat{\mathcal{A}}$ can be identified with the finite set $\mathbb{T}_N := \{\lambda \in \mathbb{C} : \lambda^N = 1\}$ (with the discrete topology).

Indeed, for $n \in \mathbb{Z}_N$, let $\delta_n \in \mathcal{A}$ be defined by $\delta_n(j) = 1$ if $j = n$ and $\delta_n(j) = 0$ otherwise. If $m, n \in \mathbb{Z}_N$, we have $\delta_m * \delta_n = \delta_k$ where $k = m + n \pmod{N}$. Further, δ_0 is the unit of $\mathcal{A}$, and $\delta_1 * \delta_1 * \dots * \delta_1$ (N -times) $= \delta_N = \delta_0$.

Assume now that $\gamma \in \widehat{\mathcal{A}}$. Then, γ is multiplicative, we get that

$$\gamma(\delta_1)^N = \gamma(\delta_1 * \delta_1 * \dots * \delta_1) = \gamma(\delta_0) = 1,$$

i.e., $\lambda := \gamma(\delta_1) \in \mathbb{T}_N$. It then follows that

$$\gamma(f) = \gamma\left(\sum_{n \in \mathbb{Z}_N} f(n)\delta_n\right) = \sum_{n \in \mathbb{Z}_N} f(n)\gamma(\delta_n) = \sum_{n \in \mathbb{Z}_N} f(n)\lambda^n$$

for all $f \in \mathcal{A}$. On the other hand, if $\lambda \in \mathbb{T}_N$, then it is not difficult to check that if $\gamma_\lambda : \mathcal{A} \rightarrow \mathbb{C}$ is given by $\gamma_\lambda(f) := \sum_{n \in \mathbb{Z}_N} f(n)\lambda^n$ for all $f \in \mathcal{A}$, then γ_λ belongs to $\widehat{\mathcal{A}}$. To see that γ_λ is multiplicative, one uses that $\lambda^N = 1$:

Indeed, let $m, n \in \mathbb{Z}_N$. Define $p \in \mathbb{Z}_N$ by $p := n - m \pmod{N}$, and note that $\lambda^p = \lambda^{n-m}$ (where $n - m \in \mathbb{Z}$ has its usual meaning):

- if $0 \leq m \leq n \leq N-1$, this is obvious;
- if $0 \leq n < m \leq N-1$, then

$$\lambda^p = \lambda^{N+n-m} = \lambda^N \lambda^{n-m} = \lambda^{n-m}.$$

This implies that $\lambda^m \lambda^p = \lambda^m \lambda^{n-m} = \lambda^n$, and the same computation as when dealing with $\mathbb{Z}$ gives that γ_λ is multiplicative.

Altogether, this shows that we may identify $\mathbb{T}_N$ with $\mathcal{A}$ via the map $\lambda \mapsto \gamma_\lambda$.

Now, setting $\omega := e^{i2\pi/N}$, we have $\mathbb{T}_N = \{\omega^0, \omega^1, \omega^2, \dots, \omega^{N-1}\}$, so we may identify $\mathbb{Z}_N$ and $\mathbb{T}_N$ via the map $m \mapsto \omega^m$ for every $m \in \mathbb{Z}_N$. Combining these maps, we may identify $\mathbb{Z}_N$ with $\mathcal{A}$ via the map $m \mapsto \gamma_{\omega^m}$.

Describe the Gelfand transform $\Gamma : \mathcal{A} \rightarrow C(\widehat{\mathcal{A}})$ as best you can.

Solution. When identifying $\widehat{\mathcal{A}}$ with $\mathbb{T}_N$ via the map $\gamma \mapsto \gamma_\lambda$, the Gelfand transform Γ of $\mathcal{A}$ becomes the map $\Gamma : \mathcal{A} \rightarrow C(\mathbb{T}_N)$ given by

$$[\Gamma(f)](\lambda) = \sum_{n \in \mathbb{Z}_N} f(n) \lambda^n, \quad f \in \mathcal{A}, \lambda \in \mathbb{T}_N,$$

that is,

$$[\Gamma(f)](\omega^m) = \sum_{n \in \mathbb{Z}_N} f(n) \omega^{mn}, \quad f \in \mathcal{A}, m \in \mathbb{Z}_N,$$

Thus, identifying $\mathbb{Z}_N$ and $\mathbb{T}_N$ as above, we get that $\Gamma : \mathcal{A} \rightarrow C(\mathbb{Z}_N)$ is given by

$$[\Gamma(f)](m) = \sum_{n \in \mathbb{Z}_N} f(n) \omega^{mn} = \sum_{n=0}^{N-1} f(n) e^{i \frac{2\pi}{N} mn}, \quad f \in \mathcal{A}, m \in \mathbb{Z}_N.$$

Note that the map $h \mapsto (h(0), h(1), \dots, h(N-1))$ is an isometric isomorphism from the Banach algebra $C(\mathbb{Z}_N)$ onto $\mathbb{C}^N$ (considered as a Banach algebra w.r.t. the $\|\cdot\|_\infty$ -norm and the pointwise product on $\mathbb{C}^N$).

On the other hand, $\mathcal{A} = \ell^1(\mathbb{Z}_N, \mathbb{C})$ can also be identified as a Banach algebra with $\mathbb{C}^N$, now considered as a Banach algebra w.r.t. the $\|\cdot\|_1$ -norm and the convolution product, via the map $f \mapsto (f(0), f(1), \dots, f(N-1))$.

Using both these identifications, the Gelfand transform of $\mathcal{A} = \ell^1(\mathbb{Z}_N, \mathbb{C})$ corresponds to the map $\Gamma : \mathbb{C}^N \rightarrow \mathbb{C}^N$ given by

$$\Gamma(z_0, z_1, \dots, z_{N-1}) = \left(\sum_{n=0}^{N-1} z_n, \sum_{n=0}^{N-1} z_n \omega^n, \sum_{n=0}^{N-1} z_n \omega^{2n}, \dots, \sum_{n=0}^{N-1} z_n \omega^{(N-1)n} \right),$$

whose standard matrix is the (unitary) matrix

$$F_N := \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \dots & \omega^{(N-1)^2} \end{bmatrix},$$

i.e.,

$$F_N = \left[e^{i \frac{2\pi jk}{N}} \right]_{j,k \in \{0,1,\dots,N-1\}}$$

which is usually called the $N \times N$ *Fourier matrix* and appears in connection with the discrete Fourier transform in numerical analysis.